Initiation à la Vérification

Binary Decision Diagrams

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Set representations

The solution of the model-checking problem for CTL and *finite-state systems* can be expressed by operations on sets:

states satisfying (sub)formulae: $[\![\psi]\!]$

subformulae combined by set operations: $\cap$, $\cup$, ...

e.g., $[\![\mathbf{EX}\,\psi]\!]$ can be obtained by the operation
$pre(S) := \{\, s \mid \exists t : s \to t \wedge t \in S \,\}$

$\mathbf{EG}$ and $\mathbf{EU}$ require fixed-point iterations on set equations

Likewise, computing the reachable states is expressible with set operations:

Start by setting $X := I$, the set of initial states;

Iterate $X := X \cup \{\, t \mid \exists s : s \to t \wedge s \in S \,\}$ until fixpoint.

Set representations

How can such sets be represented:

explicit list: $S = \{s_1, s_2, s_4, \ldots\}$

symbolic representation: compact notation or data structure

Idea: Find a data structure that

can compress the representation of large state sets

permits efficient operations that match the set operations needed in CTL model checking

Caveat

Due to the pigeon-hole principle, no lossless compression method can compress *all* sets (or work efficiently for all).

The idea that we study (binary decision diagrams) usually works well for systems whose states can be represented as Boolean vectors, with logical operations between them. We assume:

$$S = \{0,1\}^m \qquad \text{for some } m \geq 1$$

Remark: In general, the elements of *any* finite set can be represented by Boolean vectors if m is chosen large enough. However, this may not be adequate in all situations.

Literature

Some pointers:

H.R. Andersen, *An Introduction to Binary Decision Diagrams*, Lecture notes, Department of Information Technology, IT University of Copenhagen

Available on WWW

Tools:

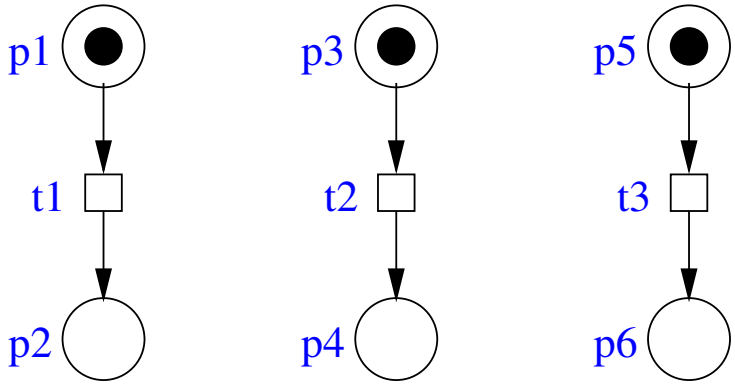
CUDD library, including DDcal (“BDD calculator”)

URL: http://vlsi.colorado.edu/~fabio/

SMV (BDD-based model checker):
http://www.cs.cmu.edu/~modelcheck/smv.html

Example 1: Petri-net

Consider the following Petri net:

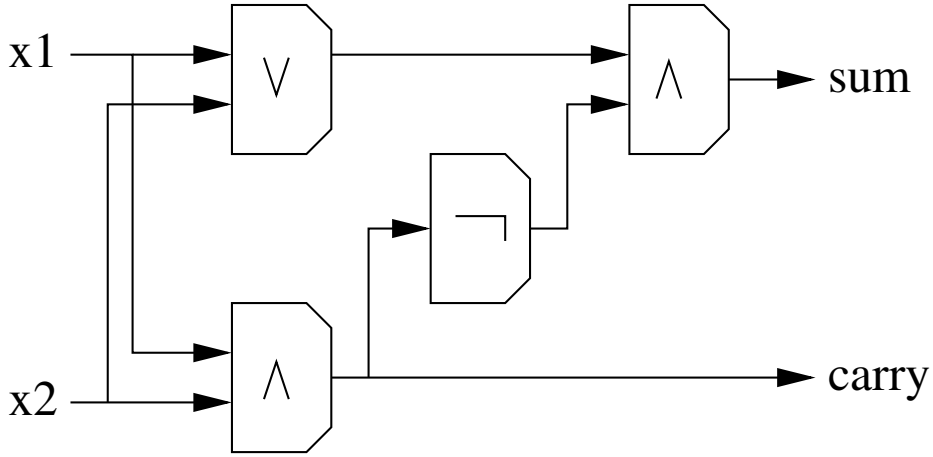


A state can be written as $(p_1, p_2, \ldots, p_6)$, where p_i, $1 \leq i \leq 6$ indicates whether there is a token on P_i.

Initial state $(1, 0, 1, 0, 1, 0)$;
other reachable states are, e.g., $(0, 1, 1, 0, 1, 0)$ or $(1, 0, 0, 1, 0, 1)$.

Example 2: Circuit

Half-adder:



The circuit has got two inputs (x_1, x_2) and two outputs (*carry*, *sum*). Their admissible combinations can be denoted by Boolean 4-tuples, e.g. $(1, 0, 0, 1)$ ($x_1 = 1$, $x_2 = 0$, *carry* $= 0$, *sum* $= 1$) is a possible combination.

The admissible combinations in Example 2 correspond to the following formula of propositional logic:

$$F \equiv \Big(carry \leftrightarrow (x_1 \wedge x_2)\Big) \wedge \Big(sum \leftrightarrow (x_1 \vee x_2) \wedge \neg carry\Big)$$

In the following, we shall treat

sets of states (i.e. sets of Boolean vectors)

and formulae of propositional logic

simply as different representations of the same objects.

Binary decision graphs

Let V be a set of variables (atomic propositions) and $<$ a total order on V, e.g.

$$x_1 < x_2 < \mathit{carry} < \mathit{sum}$$

A binary decision graph (w.r.t. $<$) is a directed, connected, acyclic graph with the following properties:

there is exactly one root, i.e. a node without incoming arcs;

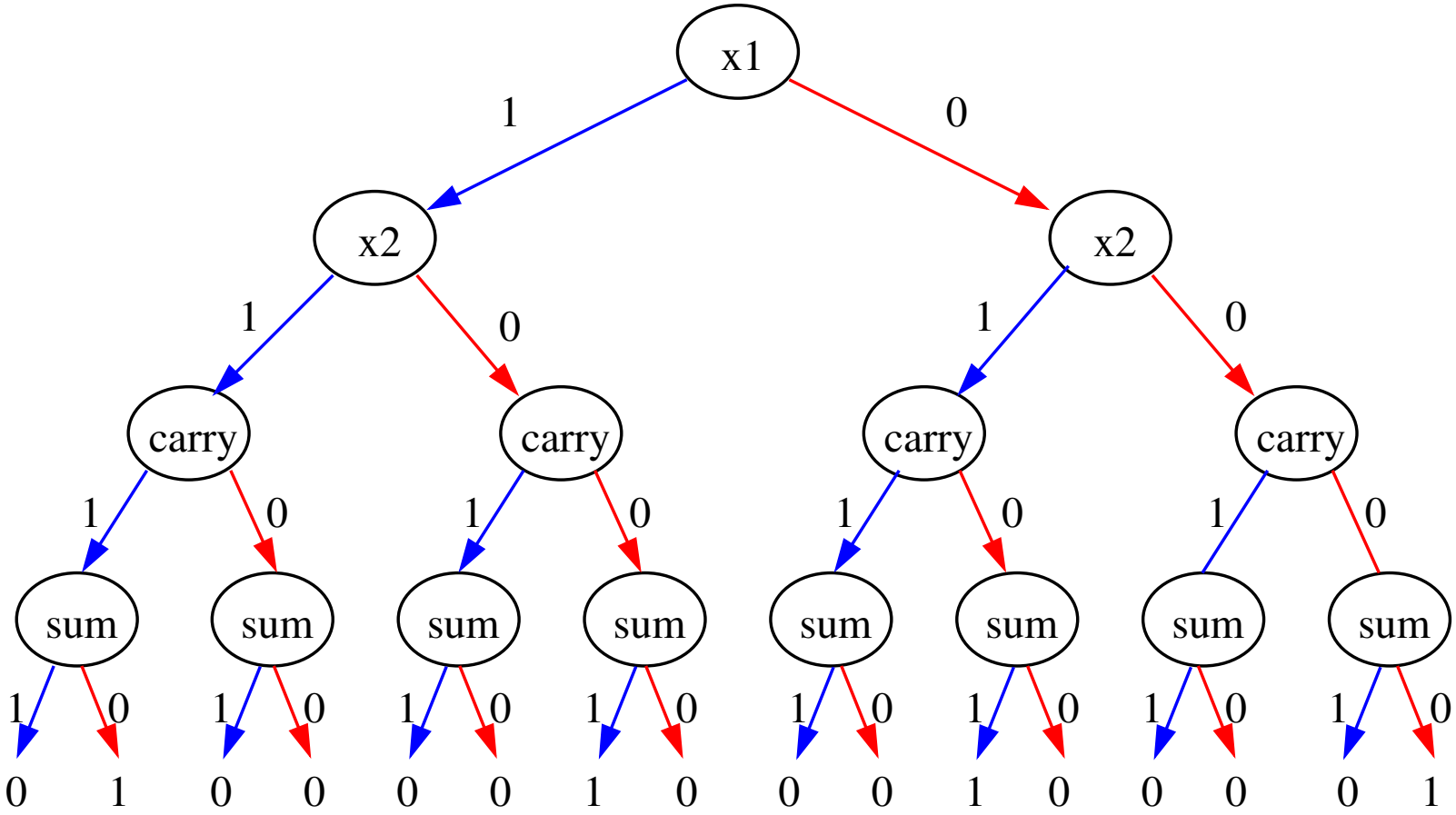
there are at most two leaves, labelled by $\mathbf{0}$ or $\mathbf{1}$;

all non-leaves are labelled with variables from V;

every non-leaf has two outgoing arcs labelled by 0 and 1;

if there is an edge from an x-labelled node to a y-labelled node, then $x < y$.

Example 2: Binary decision graph (here: a full tree)



Paths ending in 1 correspond to vectors whose entry in the truth table is 1.

Binary decision diagrams

A binary decision diagram (BDD) is a binary decision graph with two additional properties:

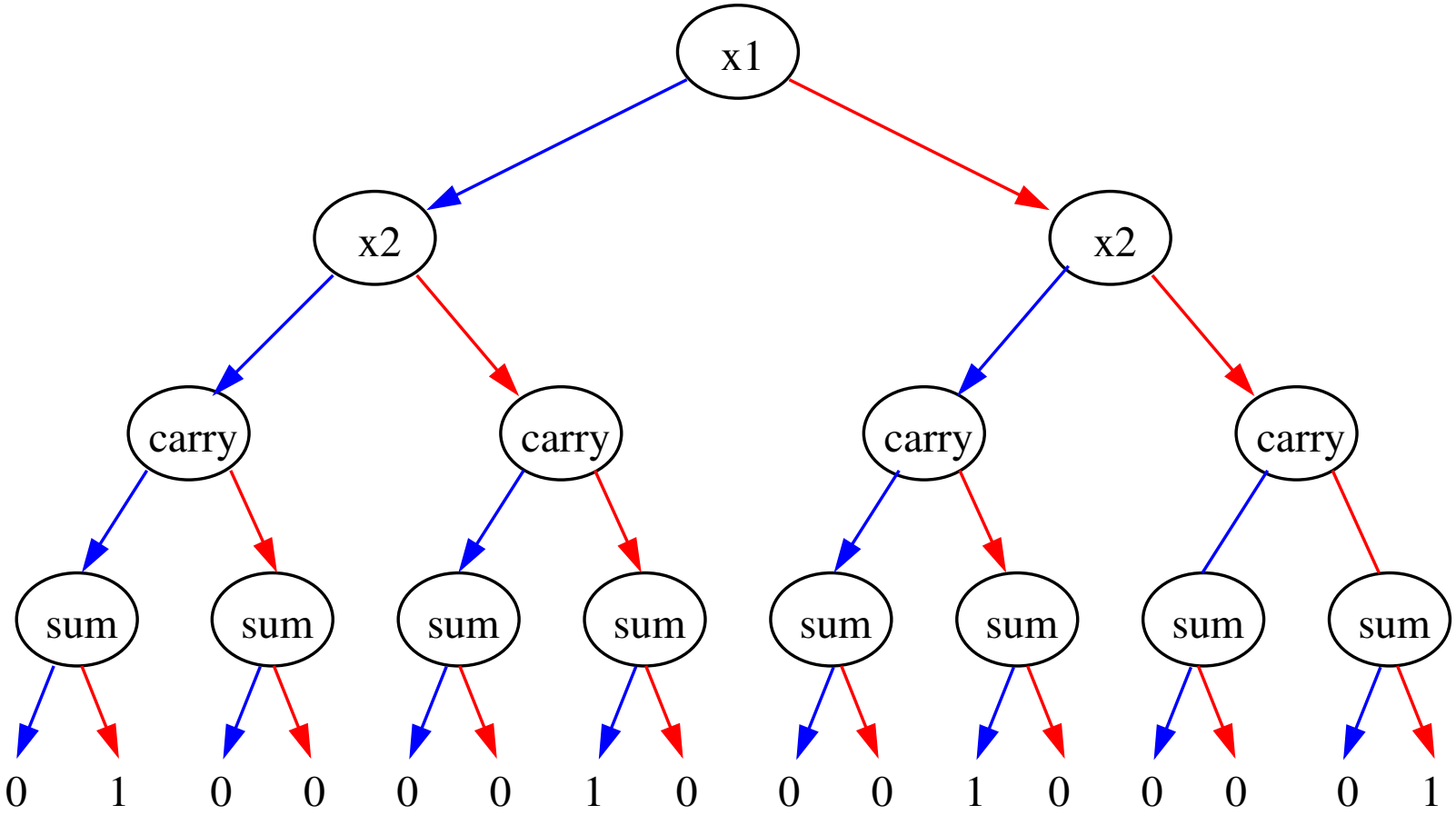
no two subgraphs are isomorphic;

there are no *redundant* nodes, where both outgoing edges lead to the same target node.

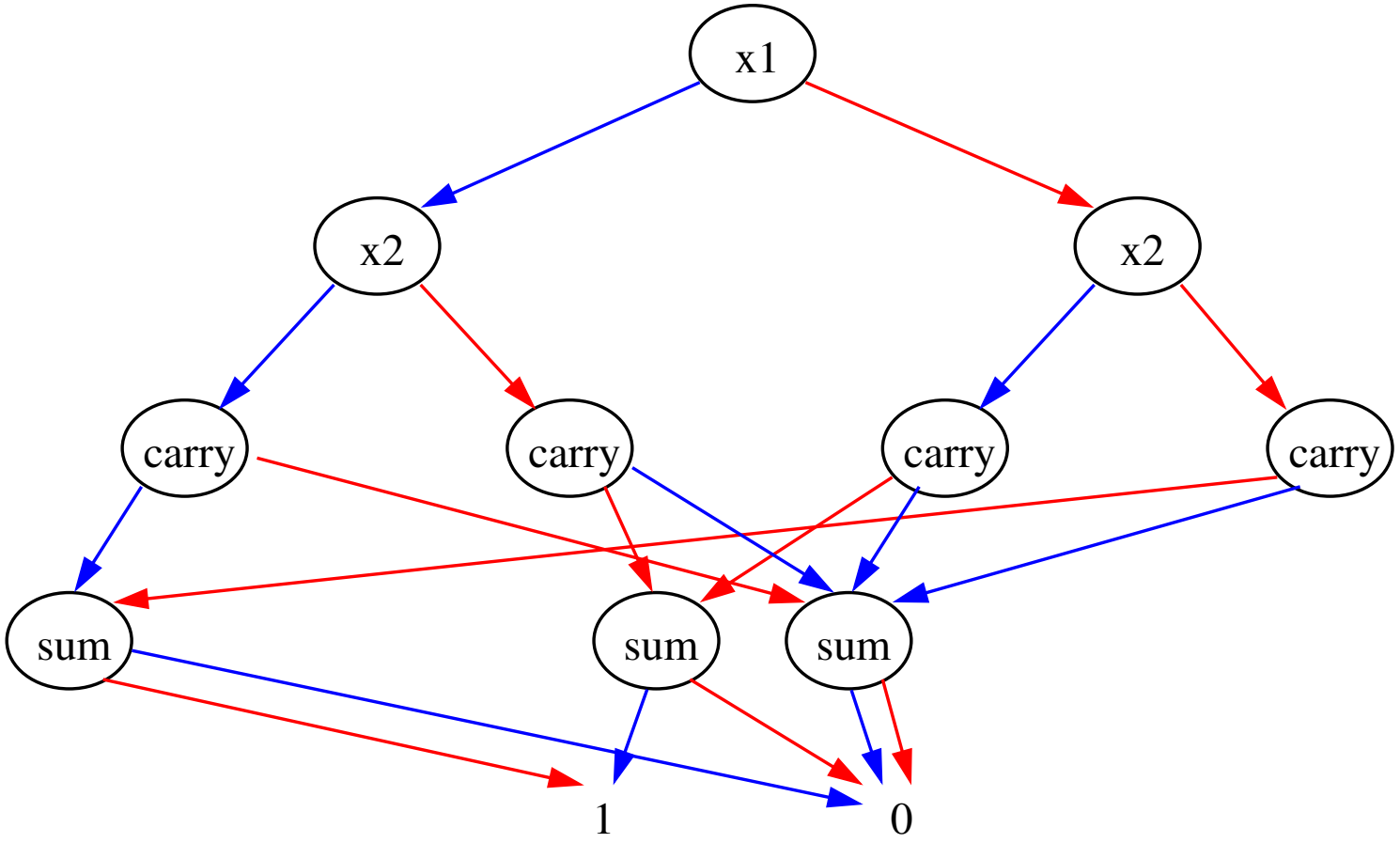
Optionally, we omit the $\mathbf{0}$-node and the edges leading there.

Remarks: On the following slides, the blue edges are meant to be labelled by 1, the red edges by 0.

Example 2: Eliminate isomorphic subgraphs (1/3)

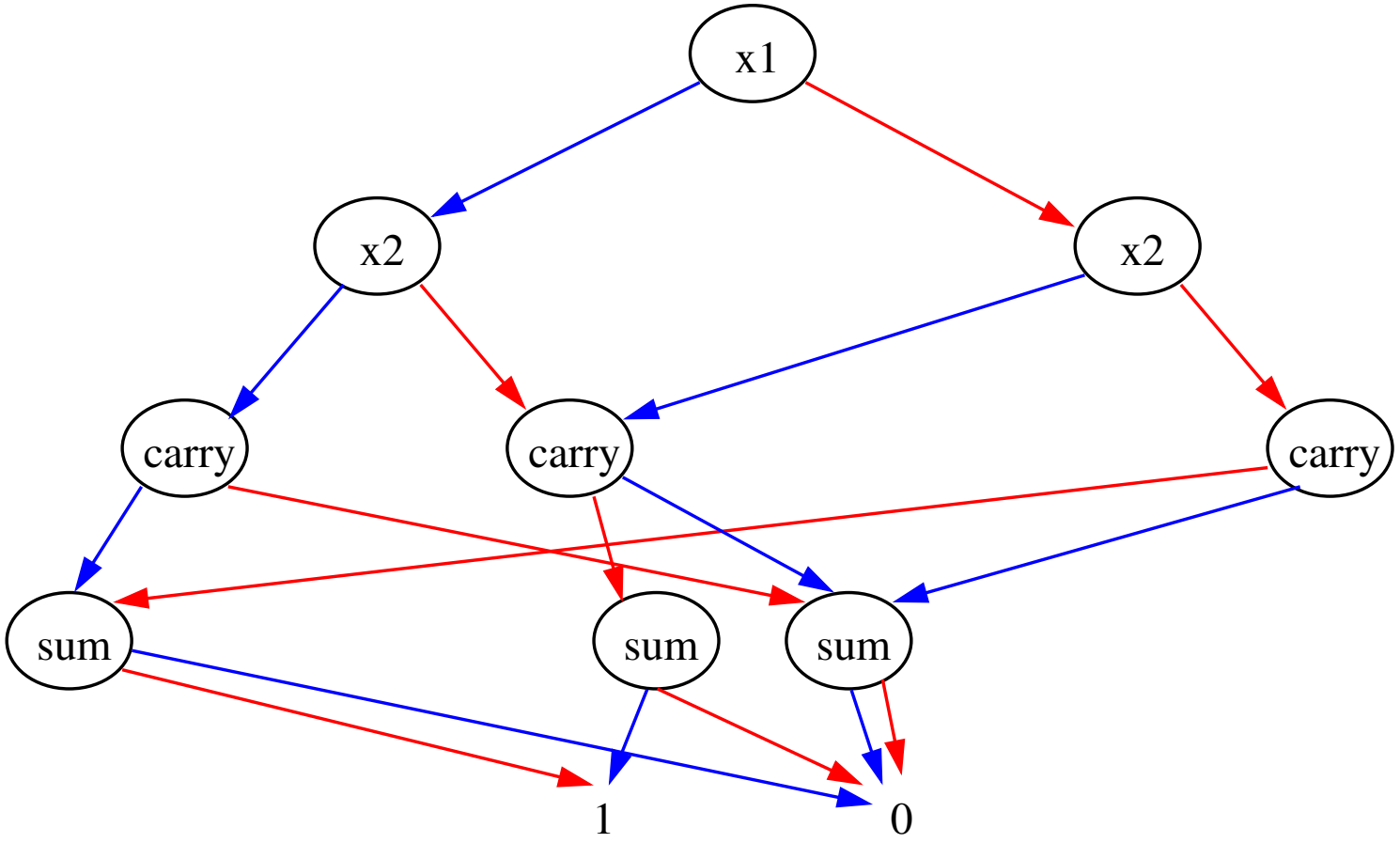


Example 2: Eliminate isomorphic subgraphs (2/3)



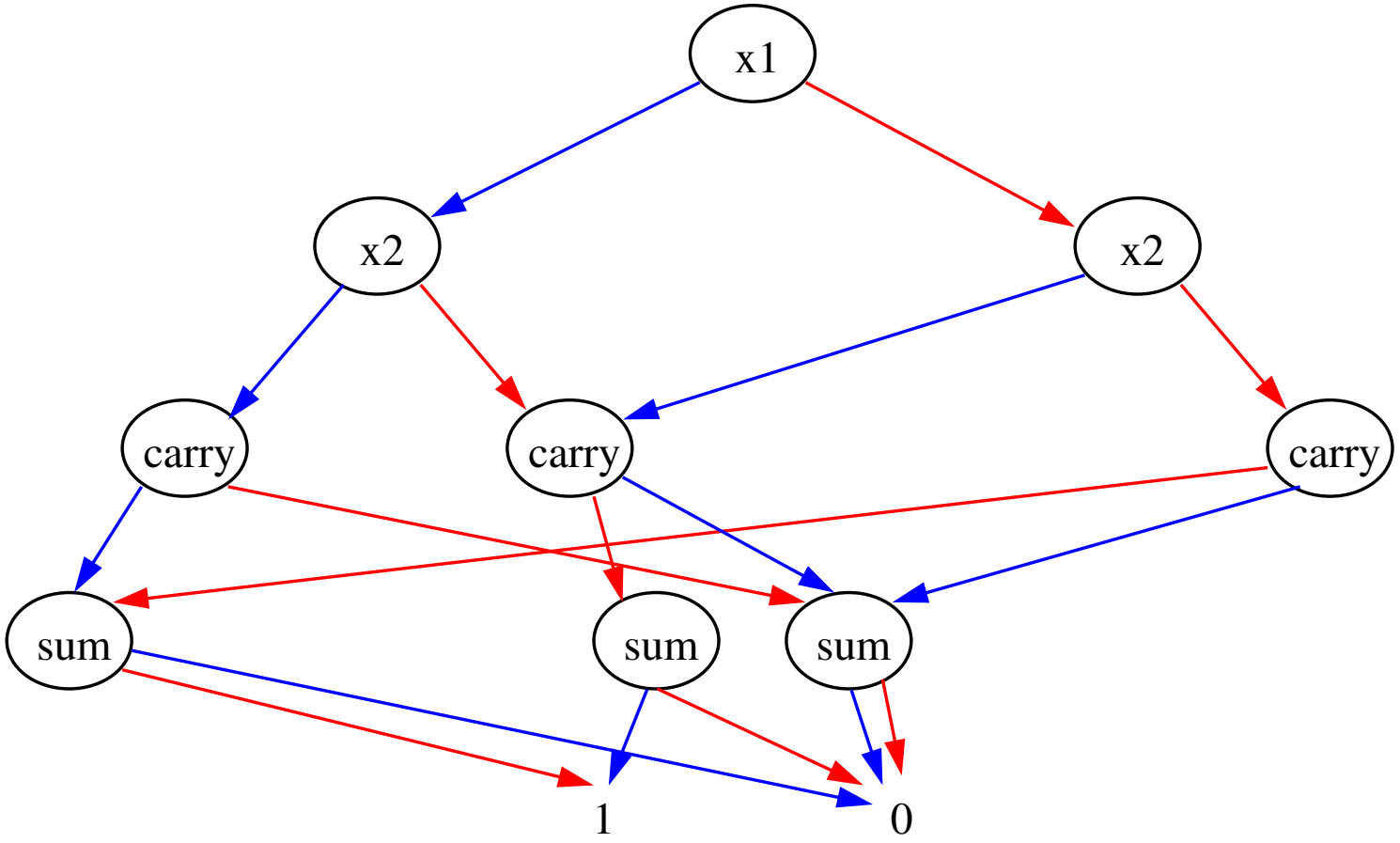
Merged the isomorphic sum-nodes (and the leaves).

Example 2: Eliminate isomorphic subgraphs (3/3)



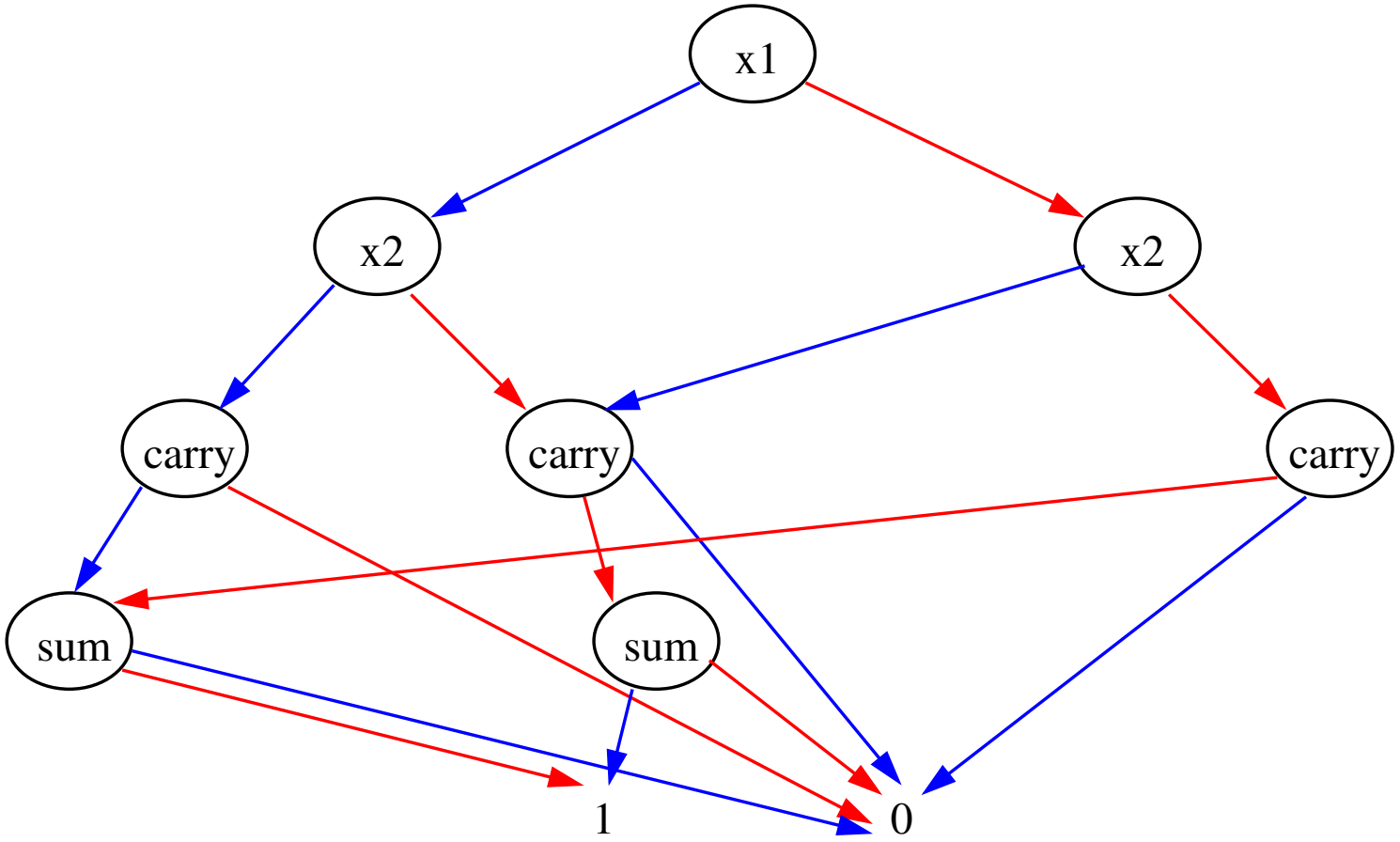
Two carry nodes can be merged, but no others $\rightarrow$ done.

Example 2: Remove redundant nodes (1/2)



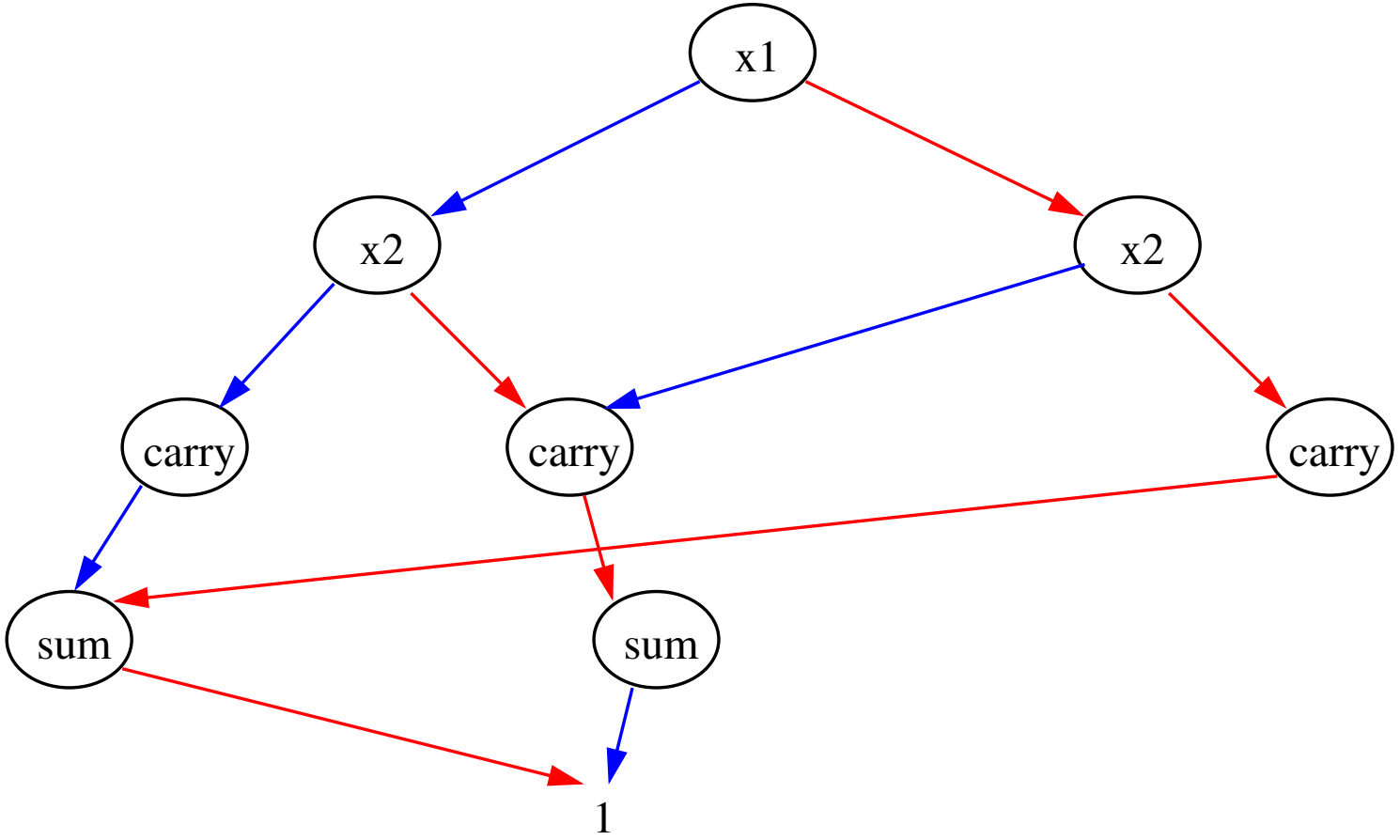
Both edges of the right *sum*-node point to 0.

Example 2: Remove redundant nodes (2/2)



No more redundant nodes $\rightarrow$ we are done.

Example 2: Omit 0-node



Optionally, we can remove the 0-node and edges leading to it, which makes the representation clearer (but still unambiguous).

Preview

In the following, we shall investigate operations on BDDs that are needed for CTL model checking.

- Construction of a BDD (from a PL formula)
- Equivalence check
- Intersection, complement, union
- Relations, computing predecessors

Propositional logic with constants

In the following, we will consider formulae of propositional logic (PL), extended with the constants 0 and 1, where:

0 is an unsatisfiable formula;

1 is a tautology.

Substitution

Let F and G be formulae of PL (possibly with constants), and let x be an atomic proposition.

$F[x/G]$ denotes the PL formula obtained by replacing each occurrence of x in F by G.

In particular, we will consider formulae of the form $F[x/0]$ and $F[x/1]$.

Example: Let $F = x \wedge y$. Then $F[x/1] = 1 \wedge y \equiv y$ and $F[x/0] = 0 \wedge y \equiv 0$.

If-then-else

Let us introduce a new, ternary PL operator. We shall call it *ite* (if-then-else).

Note: *ite* does not extend the expressiveness of PL, it is simply a convenient shorthand notation.

Let F, G, H be PL formulae. We define

$$ite(F, G, H) := (F \wedge G) \vee (\neg F \wedge H).$$

The set of INF formulae (if-then-else normal form) is inductively defined as follows:

- $\mathbf{0}$ and $\mathbf{1}$ are INF formulae;
- if x is an atomic proposition and G, H are INF formulae, then $ite(x, G, H)$ is an INF formula.

Shannon partitioning

Let F be a PL formula and x an atomic proposition. We have:

$$F \equiv \mathit{ite}(x, F[x/1], F[x/0])$$

Proof: In the following, G denotes the right-hand side of the equivalence above. Let ν be a valuation s.t. $\nu \models F$. Either $\nu(x) = 1$, then ν is also a model of $F[x/1]$ and of x and therefore also of G. The case $\nu(x) = 0$ is analogous. For the other direction, suppose $\nu \models G$. Then either $\nu(x) = 1$ and the "rest" of ν is a model of $F[x/1]$. Then, however, ν will be a model for any formula in which some of the ones in $F[x/1]$ are replaced by x, in particular also for F. The case $\nu(x) = 0$ is again analogous.

Remark: G is called the Shannon partitioning of F.

Corollary: Every PL formula is equivalent to an INF formula.
(Proof: apply the above partitioning multiple times.)

Construction of BDDs

We can now solve our first BDD-related problem: Given a PL formula F and some ordering of variables $<$, construct a BDD w.r.t. $<$ that represents F.

If F does not contain any atomic propositions at all, then either $F \equiv \mathbf{0}$ or $F \equiv \mathbf{1}$, and the corresponding BDD is simply the corresponding leaf node.

Otherwise, let x be the smallest variable (w.r.t. $<$) occurring in F. Construct BDDs B_0 and B_1 for $F[x/1]$ and $F[x/0]$, respectively (these formulae have one variable less than F).

Because of the Shannon partitioning, F is representable by a binary decision *graph* whose root is labelled by x and whose subtrees are B_0 and B_1. To obtain a BDD, we check whether B_0 and B_1 are isomorphic; if yes, then F is represented by B_0. Otherwise we merge all isomorphic subtrees in B_0 and B_1.

BDDs are unique

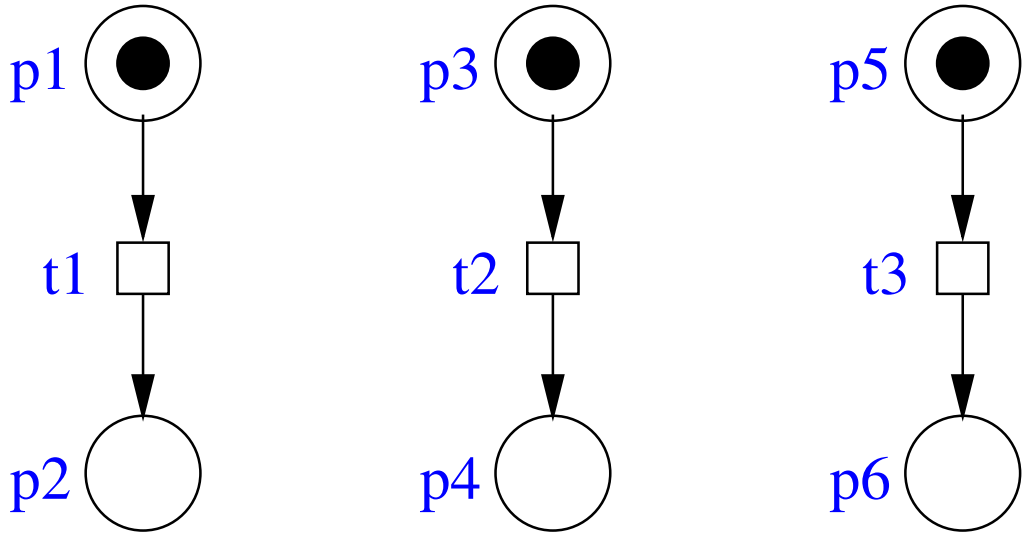
Given a PL formula F and a variable ordering $<$, there is (up to isomorphism) exactly one BDD that respects $<$ and represents F.

Proof: (sketch) by induction on the number of variables, start with 0 (constant functions), then use Shannon partitioning.

Remark: Different orderings still lead to different BDDs.
(possibly with vastly different sizes!)

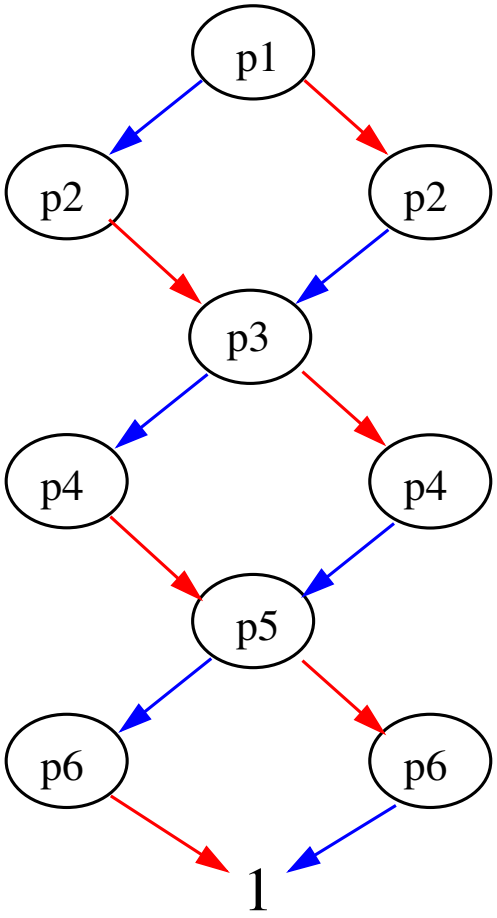
Example: Variable orderings

Recall Example 1 (the Petri net), and let us construct a BDD representing the reachable markings:



Remark: P_1 is marked iff P_2 is not, etc.

The corresponding BDD for the ordering $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$:



Remarks:

If we increase the number of components from 3 to n (for some $n \geq 0$), the size of the corresponding BDD will be linear in n.

In other words, a BDD of size n can represent 2^n (or even more) valuations.

However, the size of a BDD strongly depends on the ordering!
Example: Repeat the previous construction for the ordering

$$p_1 < p_3 < p_5 < p_2 < p_4 < p_6.$$

Equivalence test

To implement CTL model checking, we need a test for equivalence between BDDs (e.g., to check the termination of a fixed-point computation).

Problem: Given BDDs B and C (w.r.t. the same ordering) do B and C represent equivalent formulae?

Solution: Test whether B and C are isomorphic.

Special cases:

Unsatisfiability test: Check if the BDD consists just of the $\mathbf{0}$ leaf.

Tautology test: Check if the BDD consists just of the $\mathbf{1}$ leaf.

Implementing BDDs with hash tables

Suppose we want to write an application in which we need to manipulate multiple BDDs.

Efficient BDDs implementations exploit the uniqueness property by storing all BDD nodes in a hash table. (Recall that each node is in fact the root of some BDD.)

Initially, the hash table has only two unique entries, the leaves $\mathbf{0}$ and $\mathbf{1}$.

Every other node is uniquely identified by the triple (x, B_0, B_1), where x is the atomic proposition labelling that node and B_0, B_1 are the subtrees of that node, represented by pointers to their respective roots.

Usually, one implements a function $mk(x, B_0, B_1)$ that checks whether the hash table already contains such a node; if yes, then the pointer to that node is returned, otherwise a new node is created.

Each BDD is then simply represented by a pointer to its root.

A multitude of BDDs is then stored as a “forest” (a DAG with multiple roots).

Problem: garbage collection (by reference counting)

Equivalence test II

Let us reconsider the equivalence-checking problem.
(Given two BDDs B and C, do B and C represent equivalent formulae?)

If B and C are stored in hash tables (as described previously), then B and C are representable by pointers to their roots.

Due to the uniqueness property, one then simply has to check whether the pointers are the same (a constant-time procedure).

Logical operations I: Complement

Let F be a PL formula and B a BDD representing F.

Problem: Compute a BDD for $\neg F$.

Solution: Exchange the two leaves of B.

(Caution: This is not quite so simple with the hash-table implementation.)

Logical operations II: Intersection/Conjunction

Let F, G be PL formulae and B, C the corresponding BDDs (with the same ordering).

Problem: Compute a BDD for $F \wedge G$ from B and C.

We have the following equivalence:

$$F \wedge G \equiv ite(x, (F \wedge G)[x/1], (F \wedge G)[x/0]) \equiv ite(x, F[x/1] \wedge G[x/1], F[x/0] \wedge G[x/0])$$

If x is the smallest variable occurring in either F or G, then $F[x/1], F[x/0], G[x/1], G[x/0]$ are either the children of the roots of B and C (or the roots themselves).

We construct a BDD for conjunction according to the following, recursive strategy:

If B and C are equal, then return B.

If either B or C are the $\mathbf{0}$ leaf, then return $\mathbf{0}$.

If either B or C are the $\mathbf{1}$ leaf, then return the other BDD.

Otherwise, compare the two variables labelling the roots of B and C, and let x be the smaller among the two (or the one labelling both).

If the root of B is labelled by x, then let B_1, B_0 be the subtrees of B; otherwise, let $B_1, B_0 := B$. We define C_1, C_0 analogously.

Apply the strategy recursively to the pairs B_1, C_1 and B_0, C_0, yielding BDDs E and F. If $E = F$, return E, otherwise $mk(x, E, F)$.

Logical operations III: Union/Disjunction

Let F, G be PL formulae and B, C the corresponding BDDs (with the same ordering).

Problem: Compute a BDD for $F \vee G$ from B and C.

Solution: Analogous to conjunction, with the rules for $\mathbf{1}$ and $\mathbf{0}$ leaves adapted accordingly.

Complexity: With dynamic programming: $\mathcal{O}(|B| \cdot |C|)$ (every pair of nodes at most once).

Computing predecesors

In the following, we derive a strategy for computing the set

$$pre(M) = \{\, s \mid \exists s' : (s, s') \in \rightarrow \ \wedge \ s' \in M \,\}.$$

Note that the relation $\rightarrow$ is a subset of $S \times S$ whereas $M \subset S$.

We represent M by a BDD with variables $y_1, \ldots, y_m$.

$\rightarrow$ will be represented by a BDD with variables $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ (states "before" and "after").

Remark: Every BDD for M is at the same time a BDD for $S \times M$!

Thus, we can rewrite $pre(M)$ as follows:

$$\{ s \mid \exists s' : (s, s') \in \rightarrow \ \cap \ (S \times M) \}$$

Then, pre reduces to the operations intersection and existential abstraction.

Existential abstraction

Existential abstraction w.r.t. an atomic proposition x is defined as follows:

$$\exists x : F \quad \equiv \quad F[x/0] \vee F[x/1]$$

I.e., $\exists x : F$ is true for those valuations that can be extended with a value for x in such a way that they become models for F.

Example: Let $F \equiv (x_1 \wedge x_2) \vee x_3$. Then

$$\exists x_1 : F \equiv F[x_1/0] \vee F[x_1/1] \equiv (x_3) \vee (x_2 \vee x_3) \equiv x_2 \vee x_3$$

By extension, we can consider existential abstraction over sets of atomic propositions (abstract from each of them in turn).

BDDs with complement arcs

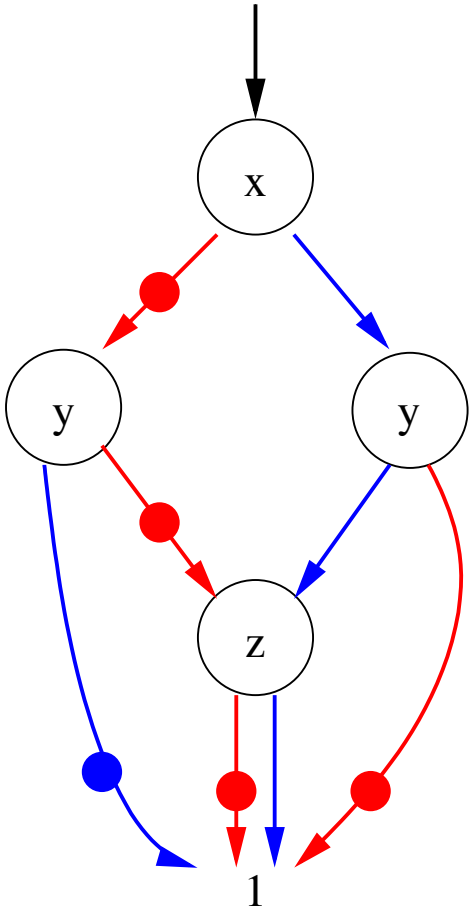
Implementation with hash tables makes negation a costly operation.

Therefore, BDD libraries often use a modification of BDDs, called BDDs with complement arcs (CBDDs).

In a CBDD, every edge is equipped with an additional bit. If the bit is true, then it means that the edge should really lead to the negation of its target.

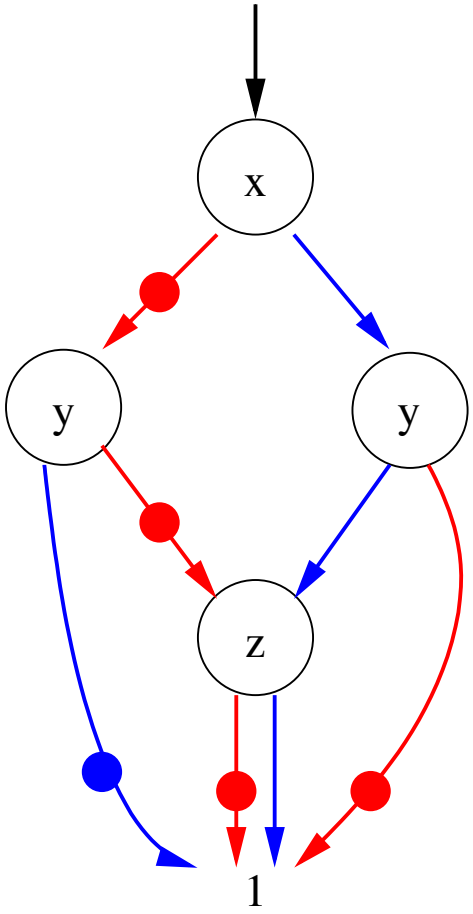
Representation: if the bit is set, we put a filled circle onto the edge.

CBDD: Example



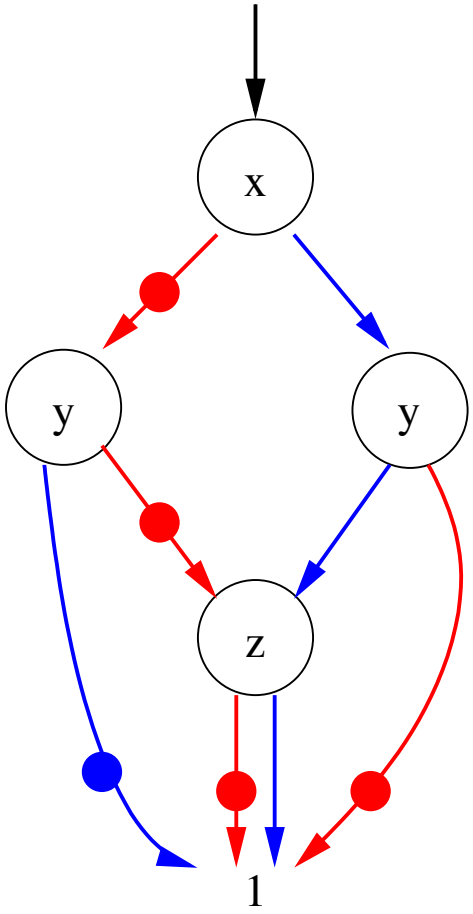
The red arc leaving the z-labelled node has its negation bit set, it therefore effectively leads to 0.

CBDD: Example



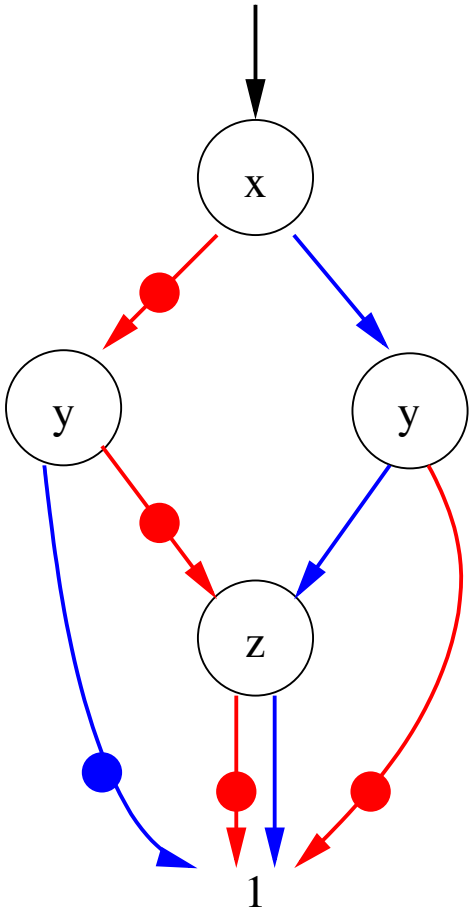
For this reason, the *z*-labelled node is *not redundant*.
The $\mathbf{0}$-leaf can be omitted altogether.

CBDD: Example



The left y-labelled node represents the formula $\neg y \wedge \neg z$.

CBDD: Example



The pointer to the root is also equipped with a negation bit (false in this case).

Remarks

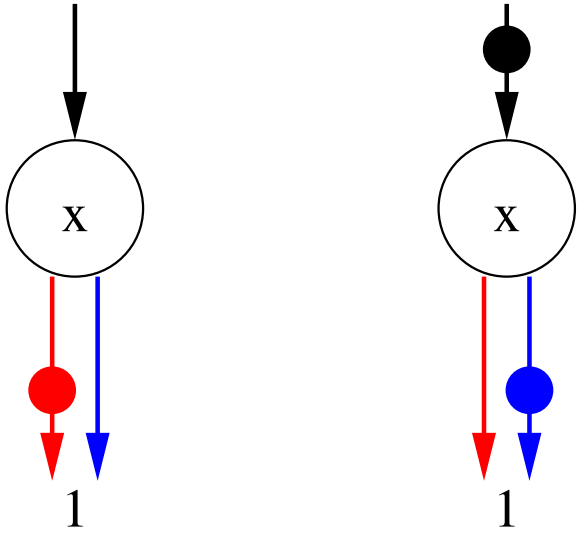
A valuation ν is a model of the formula represented by a CBDD iff the number of negations on the path corresponding to ν is *even* (including the pointer to the root).

Negation with CBDDs: trivial, invert the negation bit of the pointer to the root (constant-time operation).

Implementation (e.g., in the CUDD library): coded into the least significant bit of the pointer

Problem: CBDDs (as presented until now) are not unique!

CBDDs are not (yet) unique



Both of the CBDDs shown above represent the formula x.

Canonical form

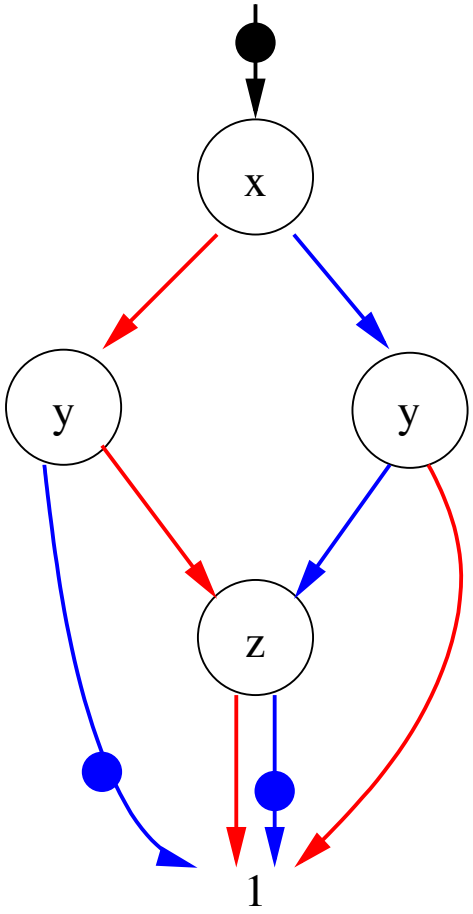
To ensure uniqueness, one can additionally prohibit negation bits on 0-labelled edges.

For this, we exploit the following equivalence:

$$ite(x, F, \neg G) \equiv \neg ite(x, \neg F, G)$$

Given any CBDD, one can eliminate negated 0-labelled edges by inverting all the negation bits on those edges that are incident with its source node (starting at the leaves, finishing with the root).

Canonical form



The CBDD shown above represents the same formula as before and does not have any negated 0-labelled edges.

LTL with BDDs

Question: Can one implement also LTL model checking using BDDs?

Answer: Yes and no (worst-case: quadratical, but works ok in practice).

Problems: BDD not compatible with depth-first search, combination with partial-order reduction difficult.

Symbolic algorithms for LTL

Idea: Find non-trivial SCCs with an accepting state, then search backwards for an initial state.

Algorithms: Emerson-Lei (EL), OWCTY

Emerson-Lei (1986)

1. Assign to M the set of all states.

2. Let $B := M \cap F$.

3. Compute the set C of states that can reach elements of B.

4. Let $M := M \cap pre(C)$.

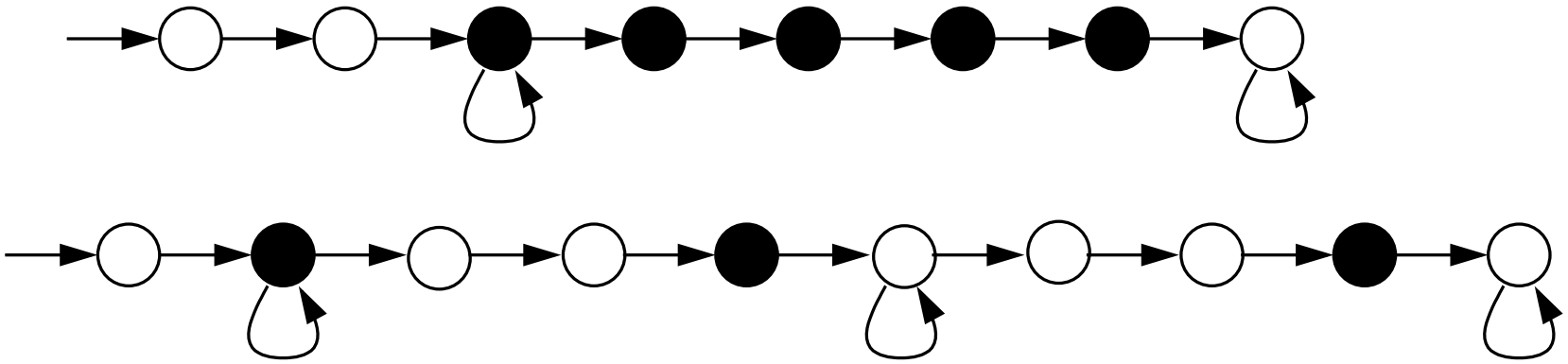
5. If M has changed, then go to step 2, otherwise stop.

One-Way-Catch-Them-Young

(Hardin et al 1997, Fisler et al 2001)

Like EL, but after step 4 repeat $M := M \cap pre(M)$ until M does not change any more.

EL and OWCTY



In the upper case, OWCTY is superior, in the lower case EL is.

In practice, OWCTY appears to work better.