Final exam for MPRI 2-9-1

20 nov. 2023 — 16h30–18h30 — no documents allowed
Réponses en français acceptées but not mandatory

1 Exercise 1: Weak WSTSes

A WSTS $(X, \leq, \to)$ is *weakly post-finite* if, and only if, for every state $x \in X$, $Post(x) \stackrel{\text{def}}{=} \{x' \in X \mid x \to x'\}$ consists of a finite union of equivalence classes with respect to $\equiv$; we write $x \equiv y$ if, and only if $x \leq y$ and $y \leq x$. It if *weakly post-effective* if, and only if, given any $x \in X$, we can compute a finite set $Post_1(x)$ of representatives of these equivalence classes; this means that we can compute a finite set $Post_1(x)$ such that $Post(x) = \{x'' \in X \mid \exists x' \in Post_1(x), x'' \equiv x'\}$. Weak post-effectiveness implies weak post-finiteness. One may write $x \to_1 x'$ to mean $x' \in Post_1(x)$.

** **Question 1.** Show that the following termination problem is decidable:

INPUT: a weakly post-effective WSTS $(X, \leq, \to)$ such that $\leq$ is decidable, a state $x_0 \in X$.

QUESTION: is every run $x_0 \to x_1 \to \cdots \to x_k \to \cdots$ starting from x_0 finite?

We give three possible solutions. The first one is the preferred one since it used theorems of the lectures instead of (re)proving variants of them.

First solution: *we define a new WSTS $(X, \leq, \to_1)$ where $x \to_1 y$ iff $y \in Post_1(x)$.*

- *Monotonicity: if $x \to_1 x'$ and $x \leq y$, then we claim that $y \to_1 y'_1$ for some $y'_1 \geq x'$. Since $x \to_1 x'$, in particular $x \to x'$. Since $(X, \leq, \to)$ is monotonic, there is a state $y' \in X$ such that $y \to y'$ and $y' \geq x'$. But it may be that $y \not\to_1 y'$. Fortunately, by definition of $Post_1$, there is a state $y'_1 \in Post_1(x')$ such that $y'_1 \equiv y'$. In particular, $y'_1 \geq x'$, and $x' \to y'_1$.*
- *Wqo: $\leq$ is wqo by assumption.*

Since $(X, \leq, \to)$ is weakly post-effective, by definition $(X, \leq, \to_1)$ is post-effective. Also, $\leq$ is decidable. Hence, by Proposition 1.36 in the lecture notes, termination is decidable for $(X, \leq, \to_1)$.

It remains to show that $(X, \leq, \to_1)$ terminates if only if $(X, \leq, \to)$ terminates. Any infinite run $x_0 \to_1 x_1 \to_1 \cdots$ is also an infinite run $x_0 \to x_1 \to \cdots$, and conversely, let us consider an infinite run $x_0 \to x_1 \to \cdots$. We will build an infinite run $x'_0 \to_1 x'_1 \to_1 \cdots$, with $x'_0 \stackrel{def}{=} x_0$ and $x'_i \geq x_i$ for every $i \in \mathbb{N}$, by induction on i. The base case is clear. Assuming $x'_i \geq x_i$, we use the fact that $(X, \leq, \to)$ is monotonic: there is a state y' such that $x'_i \to y'$ and $y' \geq x_{i+1}$. By definition of $Post_1$, there is a state $x'_{i+1} \equiv y'$ (hence $x'_{i+1} \geq x_{i+1}$) such that $x'_i \to_1 x'_{i+1}$.

Second solution. *We imitate the* proof *of Proposition 1.36. There are two semi-algorithms. The first one attempts to prove termination and builds a reachability tree $\mathcal{T}$*

from x_0*,* using only $\rightarrow_1$ transitions. *If* $\mathcal{T}$ *is finite, then* $(X, \leq, \rightarrow)$ *terminates starting from* x_0*: otherwise, as above any infinite run* $x_0 \rightarrow x_1 \rightarrow \cdots$ *gives rise to an infinite run* $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots$*. Conversely, if* $(X, \leq, \rightarrow)$ *terminates starting from* x_0*, then there is no infinite run* $x_0 \rightarrow x_1 \rightarrow \cdots$*, hence no infinite run* $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots$*; then* $\mathcal{T}$ *has only finite branches.* $\mathcal{T}$ *is finitely branching (by weak post-finiteness), hence is finite by König's Lemma, and therefore can be built (using weak post-effectiveness) in finite time.*

The other semi-algorithm enumerates all possible finite runs $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots \rightarrow_1 x_j$ *and checks whether* $x_i \leq x_j$ *for some* $0 \leq i < j$*; this is as in Lemma 1.35, but with* $\rightarrow_1$ *instead of* $\rightarrow$*. If* $(X, \leq, \rightarrow)$ *does not terminate, then there is an infinite run* $x'_0 \rightarrow x'_1 \rightarrow \cdots$ *with* $x'_0 = x_0$*. Hence, as in the First solution there is an infinite run* $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots$ *with* $x_i \geq x'_i$ *for every* $i \in \mathbb{N}$*. Since* $\leq$ *is a wqo, there must be a pair of indices* $i < j$ *such that* $x_i \leq x_j$*, hence a finite run as sought by this (second) semi-algorithm.*

Conversely, if there is a finite run $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots \rightarrow_1 x_j$ *with* $x_i \leq x_j$ *for some* $0 \leq i < j$*, then in particular* $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_j$*, and Lemma 1.35 then tells us that* $(X, \leq, \rightarrow)$ *does not terminate starting from* x_0*.*

Third solution. *As with the second solution, but building directly a finite tree as in the lectures. Its root is labeled by* x_0*, and the tree is built by adding new* $\rightarrow_1$*-successors* x_j *to existing vertices (on a branch* $x_0 \rightarrow_1 x_1 \rightarrow_1 \cdots \rightarrow_1 x_{j-1}$*) until* $x_i \leq x_j$ *for some* $0 \leq i < j$*. The proof is as with the second solution, or we can recognize that we are just replaying the algorithm of Proposition 1.36 on* $(X, \leq, \rightarrow_1)$*, and that it would perhaps be more direct to check whether* $(X, \leq, \rightarrow_1)$ *is a WSTS, which leads naturally to the first solution.*

2 Exercise 2: Sparse vectors

For a dimension $d \in \mathbb{N}$, we consider $(\mathbb{Z}^d, \leq_{\rm sp})$, the "sparser than" ordering on d-tuples of integers. This binary relation is defined via

$$\boldsymbol{a} = \langle a_1, \ldots, a_d \rangle \leq_{\rm sp} \boldsymbol{b} = \langle b_1, \ldots, b_d \rangle \overset{\text{def}}{\Leftrightarrow} \forall i, j \in \{1, \ldots, d\} \begin{cases} a_i \leq a_j \text{ if, and only if, } b_i \leq b_j \\ \text{and} \\ |a_i - a_j| \leq |b_i - b_j| \,. \end{cases}$$

Question 1. Show that $(\mathbb{Z}^d, \leq_{\rm sp})$ is a wqo.

To show that $(\mathbb{Z}^d, \leq_{sp})$ *is a wqo we associate, with any vector* $\boldsymbol{a} \in \mathbb{Z}^d$*, Boolean values* $B(\boldsymbol{a})_{i,j}$ *defined, for each* i, j *in* $\{1, \ldots, k\}$*, by* $B(\boldsymbol{a})_{i,j} \overset{def}{=}$ ***true*** *iff* $a_i \leq a_j$*,* $\overset{def}{=}$ ***false*** *otherwise, and natural numbers* $dist(\boldsymbol{a})_{i,j}$ *defined with* $dist(\boldsymbol{a})_{i,j} \overset{def}{=} |a_i - a_j|$*. Writing* $\mathbb{B}$ *for* $\{$***true***, ***false***$\}$*, the definition of* $\leq_{sp}$ *can be reformulated as*

$$\boldsymbol{a} \leq_{sp} \boldsymbol{b} \textit{ iff } B(\boldsymbol{a}) = B(\boldsymbol{b}) \wedge dist(\boldsymbol{a}) \leq_\times dist(\boldsymbol{b}) \,,$$

where $B(\boldsymbol{a})$ *and* $B(\boldsymbol{b})$ *belong to* $\mathbb{B}^{k^2}$ *(a finite set), and where* $dist(\boldsymbol{a})$ *and* $dist(\boldsymbol{b})$ *belong to* $(\mathbb{N}^{k^2}, \leq_\times)$*, a known wqo. This provides an embedding*

$$(\mathbb{Z}^d, \leq_{sp}) \quad \rightarrow \quad (\mathbb{B}^{d^2}, =) \times (\mathbb{N}^{d^2}, \leq_\times)$$

showing that $(\mathbb{Z}^d, \leq_{sp})$ *is a wqo.*

A well partial order (a wpo) is a wqo $(A, \leq_A)$ with an antisymmetric ordering $\leq_A$. A wqo is *total* (or *"linear"*) when any two elements are ordered by $\leq_A$ (there are no pairs of incomparable elements).

Question 2. For which d is $(\mathbb{Z}^d, \leq_{\text{sp}})$ a wpo? For which d is it total?

For any $x \in \mathbb{Z}$, $\boldsymbol{a} = \langle a_1, \ldots, a_d \rangle \equiv_{sp} \boldsymbol{a} + x \stackrel{def}{=} \langle a_1 + x, \ldots, a_d + x \rangle$. We say that $\boldsymbol{a} + x$ is a shift *of $\boldsymbol{a}$. In fact, $\boldsymbol{a} \equiv_{sp} \boldsymbol{b}$ iff $\boldsymbol{a}$ is a shift of $\boldsymbol{b}$ and the equivalence class $[\boldsymbol{a}]_{\equiv_{sp}}$ contains exactly all the shifts of $\boldsymbol{a}$.*

Thus in $(\mathbb{Z}^1, \leq_{sp})$ all elements are equivalent w.r.t. $\equiv_{sp}$. In general, an equivalence class $[\boldsymbol{a}]_{\equiv_{sp}}$ is infinite since it consists of all $\boldsymbol{a} + x$, unless $d = 0$ in which case we have only one element, the empty tuple $\langle\rangle$, that satisfies $\langle\rangle = \langle\rangle + x$ for any $x \in \mathbb{Z}$.

Thus $(\mathbb{Z}^d, \leq_{sp})$ is antisymmetric, hence wpo, only when $d = 0$. It is linear only when $d \leq 1$: for $d \geq 2$ the elements $\langle 1, 0, \ldots \rangle$ and $\langle 0, 1, \ldots \rangle$ are incomparable.

The equivalence induced by $\leq_{\text{sp}}$ is written $\equiv_{\text{sp}}$ and defined via $\boldsymbol{a} \equiv_{\text{sp}} \boldsymbol{b} \stackrel{\text{def}}{\Leftrightarrow} \boldsymbol{a} \leq_{\text{sp}} \boldsymbol{b} \wedge \boldsymbol{b} \leq_{\text{sp}} \boldsymbol{a}$. As usual, we may quotient $(\mathbb{Z}^d, \leq_{\text{sp}})$ by the induced equivalence and obtain a wpo.

Question 3. Give a simple description (e.g., modulo isomorphism) of the quotient wpo $(\mathbb{Z}^2, \leq_{\text{sp}})/_{\equiv_{\text{sp}}}$.

For $(\mathbb{Z}^2, \leq_{sp})$ we can define the canonical representative of a pair $\langle a_1, a_2 \rangle$ as the pair $\langle 0, a_2 - a_1 \rangle$ obtained by fixing the fist component to zero. For these canonical representatives one has

$$\langle 0, a \rangle \leq_{sp} \langle 0, b \rangle \stackrel{def}{\Leftrightarrow} |a| \leq |b| \wedge sign(a) = sign(b) \,,$$

where $sign(a) \in \{+, 0, -\}$ is $+$ if $a > 0$, is $-$ if $a < 0$ and is 0 when $a = 0$. Thus $\langle 0, 1 \rangle$, $\langle 0, 0 \rangle$ and $\langle 0, -1 \rangle$ are incomparable.

The quotient $(\mathbb{Z}^2, \leq_{sp})/_{\equiv_{sp}}$ is then isomorphic to $\omega \sqcup \Gamma_1 \sqcup \omega$.

Question 4. Give an antichain of size n in $(\mathbb{Z}^3, \leq_{\text{sp}})$.

A possible answer is $\langle 0, 0, n \rangle$ $\langle 0, 1, n-1 \rangle$ $\langle 0, 2, n-2 \rangle$ $\cdots$ $\langle 0, n-1, 1 \rangle$.

We consider the norm on $\mathbb{Z}^d$ given by $|\langle a_1, \ldots, a_d \rangle| \stackrel{\text{def}}{=} \max(|a_1|, \ldots, |a_d|)$. This turns $(\mathbb{Z}^d, \leq_{\text{sp}})$ into a normed wqo, or nwqo, and lets us define the length function $L_{g,(\mathbb{Z}^d, \leq_{\text{sp}})}$. In the following we fix $g(n) = n + 1$ and simply write L_d for the corresponding length function.

** **Question 5.** Give a simple numerical expression for $L_2(n)$, i.e., for $d = 2$. A formal proof is not required here, only a correct expression with a few lines of justification.

One has $L_2(n) = 8n + 9$. The longest (g, n)-controlled bad sequence is

$$\langle 0,0 \rangle\ \langle n+1, -n-1 \rangle\ \langle n+1, -n \rangle\ \langle n, -n \rangle\ \langle n, 1-n \rangle\ \langle n-1, 1-n \rangle\ \langle n-1, 2-n \rangle\ \cdots\ \langle 1, -1 \rangle\ \langle 1, 0 \rangle$$
$$\langle -3n-3, 3n+3 \rangle\ \langle -3n-3, 3n+2 \rangle\ \langle -3n-2, 3n+2 \rangle\ \langle -3n-2, 3n+1 \rangle\ \cdots\ \langle -1, 1 \rangle\ \langle -1, 0 \rangle.$$

Here, when we pick $\langle n+1, -n-1 \rangle$ and later $\langle -3n-3, 3n+3 \rangle$ we pick a $\leq_{sp}$-maximal element in the residual associated with the previous elements in the bad sequence.

3 Exercise 3: Ordering ω-words

Let $(A, \leq)$ be a wqo. The set A^* of finite words over A is ordered by the subword ordering $\leq_*$ as seen in class. We aim to extend this to infinite words, called *ω-words* since they are infinite to the right. Now for two ω-words $v = (x_i)_{i\in\mathbb{N}}$ and $w = (y_i)_{i\in\mathbb{N}}$ in A^ω, we define

$$v \leq_\omega w \overset{\text{def}}{\Leftrightarrow} \begin{cases} \text{there are some indexes } n_0 < n_1 < n_2 < \cdots \\ \text{such that } x_i \leq y_{n_i} \text{ for all } i \in \mathbb{N}. \end{cases}$$

Since $(A, \leq)$ is a quasi-order, $\leq_\omega$ is reflexive and transitive, hence $(A^\omega, \leq_\omega)$ is a quasi-order.

We start with the ω-word extension of $(\mathbb{N}, \leq)$ and consider ω-words $v, w \in \mathbb{N}^\omega$ of natural numbers. We say that an ω-word $v \in \mathbb{N}^\omega$ is *unbounded* if it contains arbitrarily large natural numbers.

Question 1. What is $\leq_\omega$ restricted to such unbounded ω-words?

If v is unbounded then $w \leq_\omega v$ for any $w \in \mathbb{N}^\omega$. All unbounded ω-words are equivalent w.r.t. $\leq_\omega$ and are a supremum for $(\mathbb{N}^\omega, \leq_\omega)$.

With a bounded ω-words $v \in \mathbb{N}^\omega$, of the form $v = x_0, x_1, x_2, \ldots$, we associate $\Lambda(v)$, defined as $\Lambda(v) \overset{\text{def}}{=} \limsup_i x_i = \lim_{k\to\infty} \max_{i\geq k} x_i$ (note that $\Lambda(v)$ is a finite number since v is bounded), and we let $M(v)$ be the first index such that $x_i \leq \Lambda(v)$ for all $i \geq M(v)$. The finite sequence $\dot{v} \overset{\text{def}}{=} x_0, \ldots, x_{M(v)-1}$ is the shortest prefix of v such that v can be written $v = \dot{v} \cdot \ddot{v}$ with $\ddot{v}$ an ω-length suffix having all its elements bounded by $\Lambda(v)$.

Question 2. Assume that $w = y_0, y_1, y_2, \ldots$ is a second bounded ω-word and show that

$$\Lambda(v) \leq \Lambda(w) \text{ implies } \ddot{v} \leq_\omega \ddot{w}\,, \tag{†}$$

$$\big(\Lambda(v) \leq \Lambda(w) \;\wedge\; \dot{v} \leq_* \dot{w}\big) \text{ implies } v \leq_\omega w\,. \tag{‡}$$

(†)*: $\Lambda(w)$ appears infinitely many times in $\ddot{w}$. This proves $\ddot{v} \leq_\omega \ddot{w}$ since every element of $\ddot{v}$ is $\leq \Lambda(v)$.*
(‡)*: by combining an embedding of $\dot{v}$ into $\dot{w}$ and an embedding of $\ddot{v}$ into $\ddot{w}$, one builds an embedding of $\dot{v}\ddot{v}$, i.e., v, into $\dot{w}\ddot{w}$, i.e., w.*

Question 3. Eq. (‡) gives a sufficient condition for $v \leq_\omega w$ on bounded ω-words. Is it a necessary condition?

No: $v \leq_\omega w$ requires $\Lambda(v) \leq \Lambda(w)$ but does not require $\dot{v} \leq_* \dot{w}$. Consider for example $v = 1, 0, 0, 0, \ldots$ and $w = 1, 1, 1, 1, \ldots$. One has:

$$\begin{array}{lllll} v = 1(0)^\omega, & \Lambda(v) = 0, & M(v) = 1, & \dot{v} = 1, & \ddot{v} = (0)^\omega, \\ w = (1)^\omega, & \Lambda(w) = 1, & M(w) = 0, & \dot{w} = \epsilon, & \ddot{w} = (1)^\omega. \end{array}$$

Clearly $v \leq_\omega w$ but one has nonetheless $\dot{v} \not\leq_* \dot{w}$.

Question 4. Show that $(\mathbb{N}^\omega, \leq_\omega)$ is a wqo.

*All the hard work has been done in the first two questions. If an infinite sequence $S = v_0, v_1, v_2, \ldots$ contains an unbounded v_i then it has an increasing pair as seen in question **Question 1.** (except if $i = 0$). Otherwise all the v_i's are bounded (perhaps after removing v_0) and we can extract an infinite subsequence with increasing $\Lambda(v_i)$ —since $(\mathbb{N}, \leq)$ is a wqo—, and further extract a pair with increasing $\dot{v}_i$'s —since $(\mathbb{N}^*, \leq_*)$ is a wqo by Higman's Lemma. With now $\Lambda(v_i) \leq \Lambda(v_j)$ and $\dot{v}_i \leq_* \dot{v}_j$, Eq. (‡) yields $v_i \leq_\omega v_j$ and S contains an increasing pair.*

Question 5. Generalise the previous question and show that $(A^\omega, \leq_\omega)$ is a wqo when $(A, \leq)$ is a *total* (or *linear*) wqo, i.e., when any two elements are comparable.

We say that an ω-word $v = x_0, x_1, x_2, \ldots \in A^\omega$ is eventually bounded in A *if there is $a \in A$ s.t. $x_i \leq a$ for almost all i. In this case, and since $(A, \leq)$ is a linear wqo, there is a smallest a that eventually dominates v, and we let $\Lambda(v) \stackrel{def}{=} \limsup_{i \in \mathbb{N}} x_i$ denote this minimal a. This leads to a factorisation $v = \dot{v}\ddot{v}$ such that $\dot{v} \in A^*$ is the shortest prefix of v containing all x_i that are strictly above $\Lambda(v)$. One can then check that if $v, w \in A^\omega$ are eventually bounded in A then Eq. (‡) is satisfied. Therefore, from an infinite sequence $S = v_0, v_1, v_2, \ldots$ where all v_i are eventually bounded, we can extract as in the previous question a pair v_i, v_j with $\Lambda(v_i) \leq \Lambda(v_j)$ and $\dot{v}_i \leq_* \dot{v}_j$, hence with $v_i \leq_\omega v_j$. There remains the case of words $v \in A^\omega$ that are not eventually bounded, but then $v \geq_\omega w$ for any w, i.e., v is a supremum in $(A^\omega, \leq_\omega)$, and any infinite sequence with such words (not in first position) contains an increasing pair.*

Question 6. We consider a finite alphabet $(\Sigma, =)$ equipped with the empty ordering. Show that its ω-word extension $(\Sigma^\omega, \leq_\omega)$ is a wqo.

With an ω-word $v \in \Sigma^\omega$ we associate $\inf(v) \stackrel{def}{=}$ the set of symbols that appear infinitely often in v, and factor v under the form $v = \dot{v}\ddot{v}$ such that $\dot{v}$ is the shortest prefix of v that contains all $x_i \notin \inf(v)$. Clearly, $\ddot{v} \leq_\omega \ddot{w}$ iff $\inf(v) \subseteq \inf(w)$, and

$$\big(\inf(v) = \inf(w) \ \wedge \ \dot{v} \leq_* \dot{w}\big) \textit{ implies } v \leq_\omega w \,.$$

One concludes by extracting from any infinite sequence $v_0, v_1, v_2, \ldots$ of ω-word a pair with $\inf(v_i) = \inf(v_j)$ and $\dot{v}_i \leq_* \dot{v}_j$. [NB: $(\Sigma, =)$ is not a linear wqo, but it is a finite union of (trivial) linear wqos. The above proof extends to all finite unions of linear wqos.]

*** **Question 7.** *This is an optional question in case you have answered everything else.* We return to the general case where $(A, \leq)$ is a wqo. Show that $(A^\omega, \leq_\omega)$ is well-founded.

I'll wait and write an answer when I have looked at all submitted exams: perhaps one will contain a nice and elegant proof?!