Labelled Well Quasi Ordered Classes of Bounded (Linear) Clique-Width

an automata based approach

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2024-11-07

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Well Quasi Orderings

Every infinite sequence of elements contains an increasing pair.

What is a well-quasi-order (WQO)?

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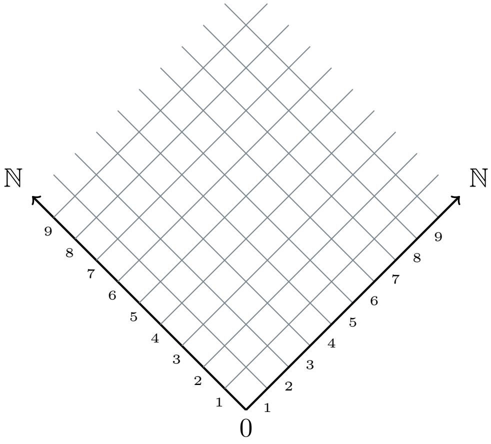

Well-Quasi-Order over $\mathbb{N} \times \mathbb{N}$

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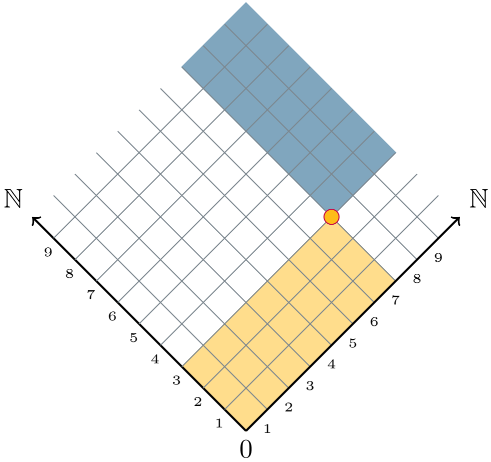

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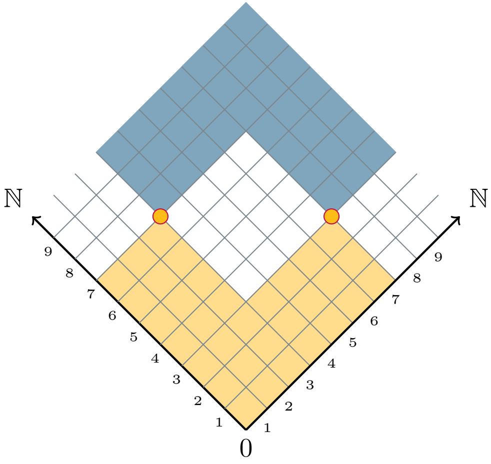

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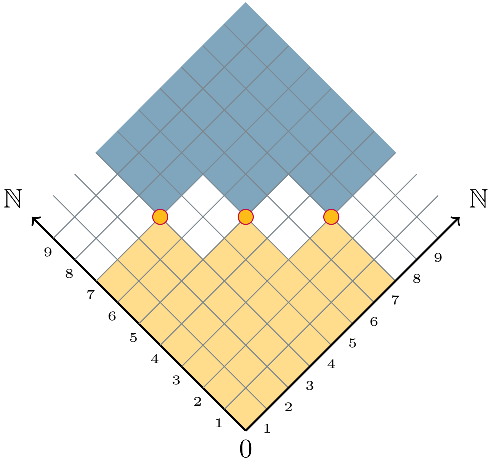

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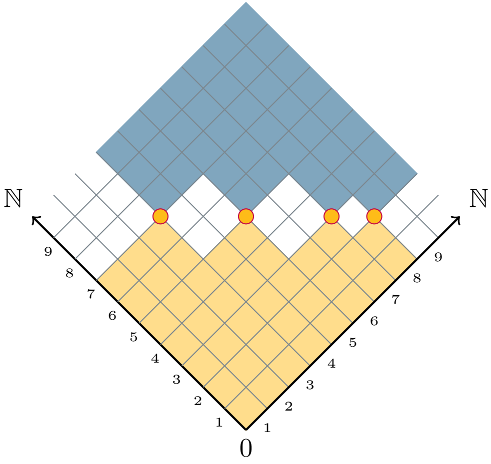

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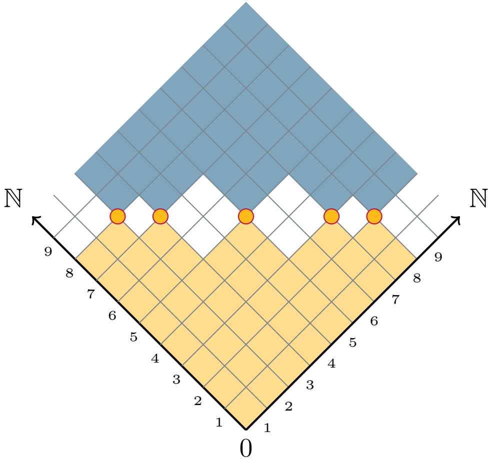

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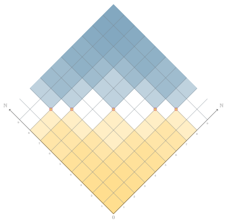

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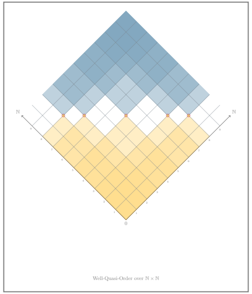

Theory **and** *Practice*

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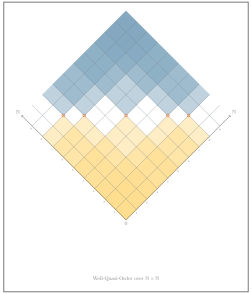

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Verification of Well-Structured-Transition-Systems [Abd+96]

Karp-Miller Coverability [KM69]

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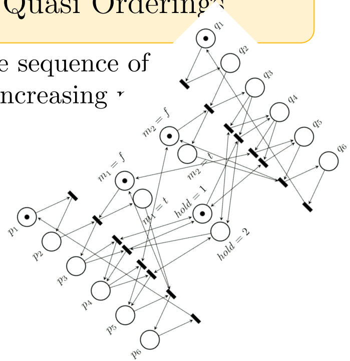

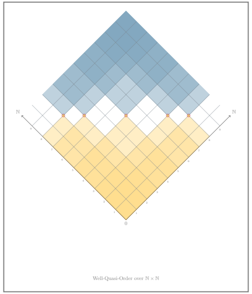

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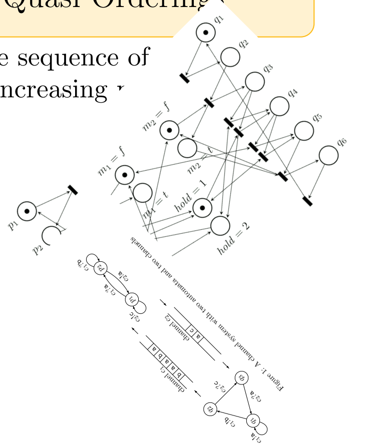

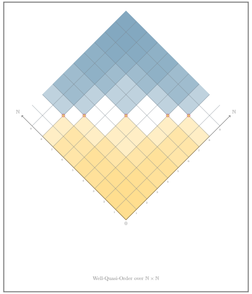

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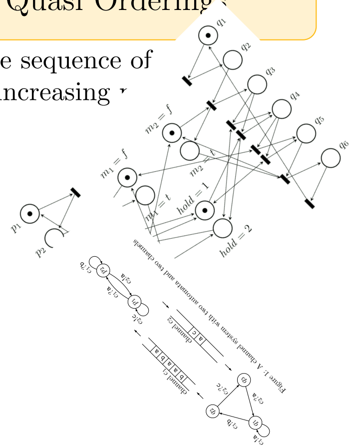

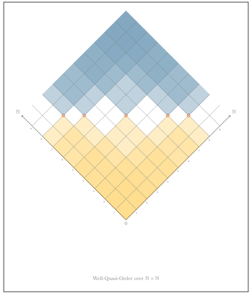

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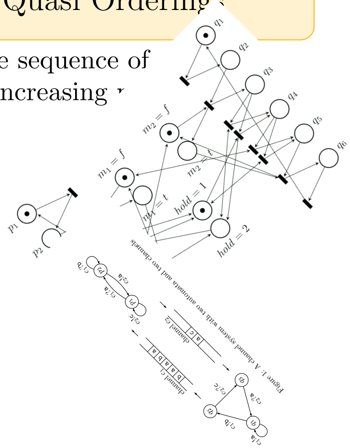

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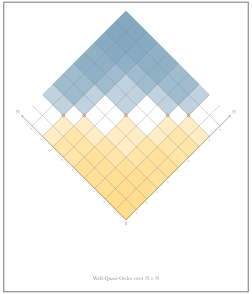

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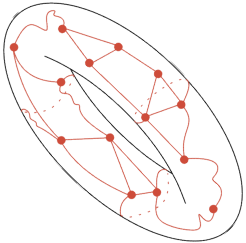

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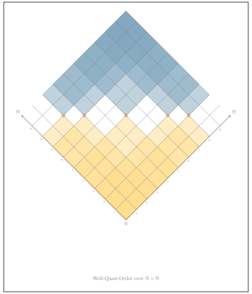

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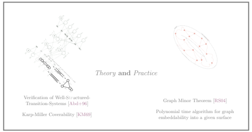

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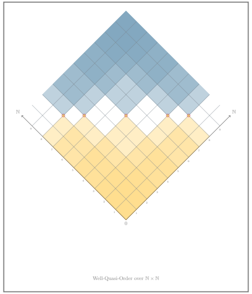

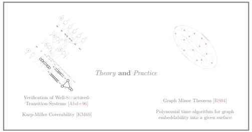

A beautiful *compositional* theory

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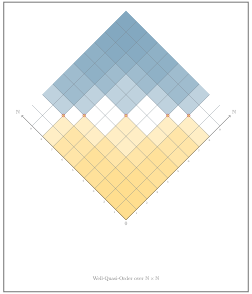

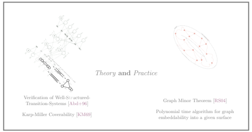

A beautiful *compositional* theory

$$(X, \leq_X) \times (Y, \leq_Y)$$

$$(X, \leq_X) + (Y, \leq_Y)$$

$$(X, \leq_X) \subseteq (Y, \leq_Y)$$

Algebraic Operations

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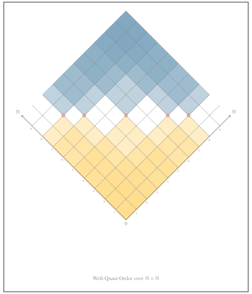

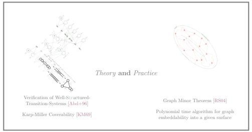

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$(\mathcal{P}_{\text{fin}}(X), \leq_{\text{hoare}})$

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Set Functors

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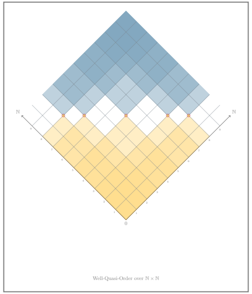

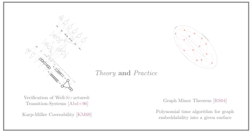

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Structural Functors

$(X^*, \leq_{\text{subword}})$ [Hig52]

$(\mathsf{Trees}(X), \leq_{\text{subtree}})$ [Kru72]

$(\mathsf{Graphs}(X), \leq_{\text{minor}})$ [RS04]

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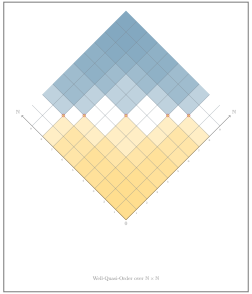

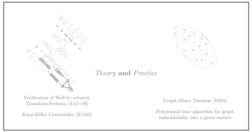

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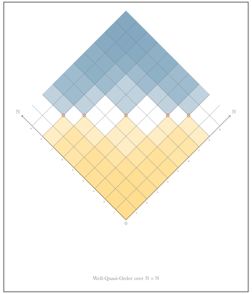

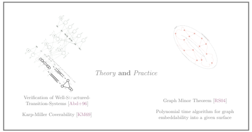

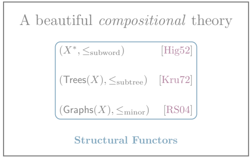

Some of these are *inductive* datatypes [Lop23; Fre20], but we want *more*! Also, can we avoid *minimal bad sequence arguments* [Nas65]?

Classes of Structures...

... as Operators

Structures as Operators

Classes of Structures...

... as Operators

Higman's Subword Embedding

m t p l

Classes of Structures...

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m o n t p e l l i e r

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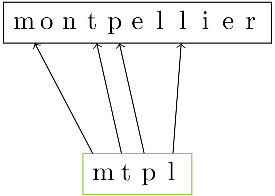
Higman's Subword Embedding



$$\sigma = \{(\leq, 2)\}$$

Classes of Structures...

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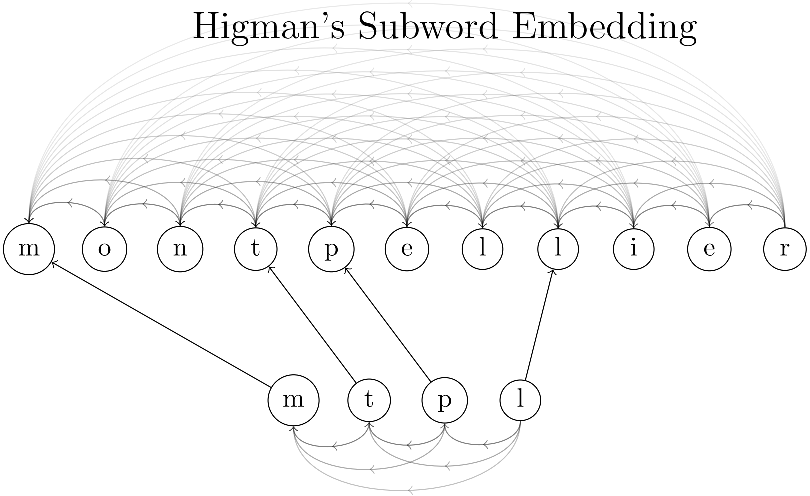
Higman's Subword Embedding



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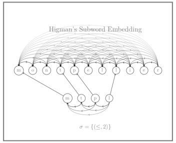

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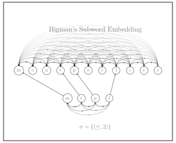
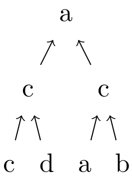


Kruskal's Tree Embedding



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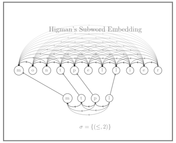

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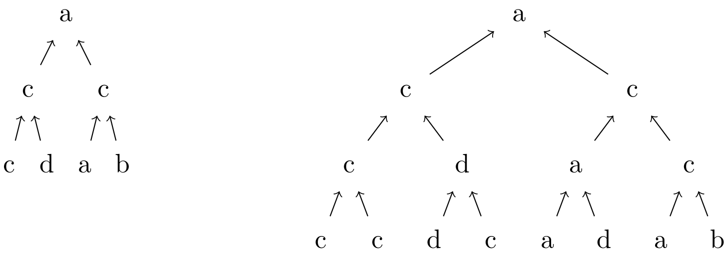

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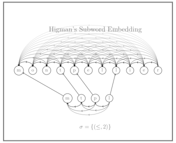

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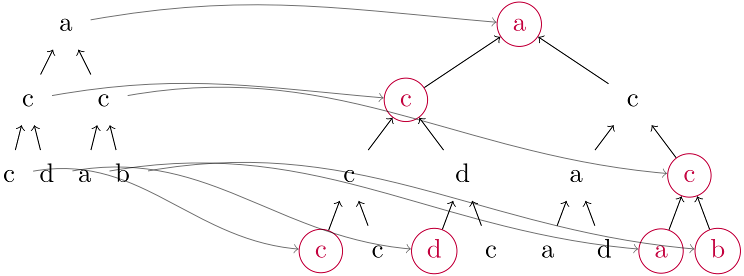

$$\sigma = \{(\leq, 2), (\text{leastCommonAncestor}, 3)\}$$

Classes of Structures...

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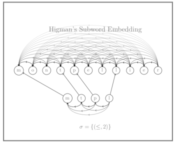

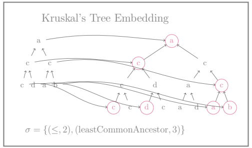

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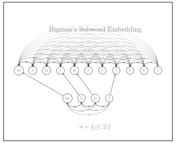

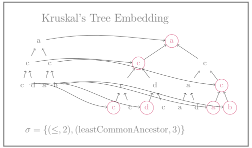

In General

$\mathcal{C}$ a class of finite structures

$\mathsf{Label}_X(\mathcal{C})$ the X-labelled structures in $\mathcal{C}$

$\mathfrak{A} \leq_X \mathfrak{B}$ if there exists a monotone embedding

$(X, \leq)$ WQO $\implies$ $(\mathsf{Label}_X(\mathcal{C}), \leq_X)$ is a WQO?

Classes of Structures...

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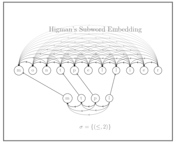
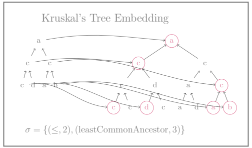




In General	This talk
$\mathcal{C}$ a class of finite structures	$\mathcal{C}$ is a class of **finite undirected graphs**
$\mathsf{Label}_X(\mathcal{C})$ the X-labelled structures in $\mathcal{C}$	$\mathsf{Label}_X(\mathcal{C})$ is a class of **X-labelled-graphs**
$\mathfrak{A} \leq_X \mathfrak{B}$ if there exists a monotone embedding	$\leq_X$ is the **induced subgraph** relation
$(X, \leq)$ WQO $\implies$ $(\mathsf{Label}_X(\mathcal{C}), \leq_X)$ is a WQO?	Is the class of graphs a WQO-operator?

Pouzet Conjectures

Graph Theory	Operators
Is $\mathcal{C}$ WQO?	Is $\mathcal{C}$ + WQO set of labels WQO?

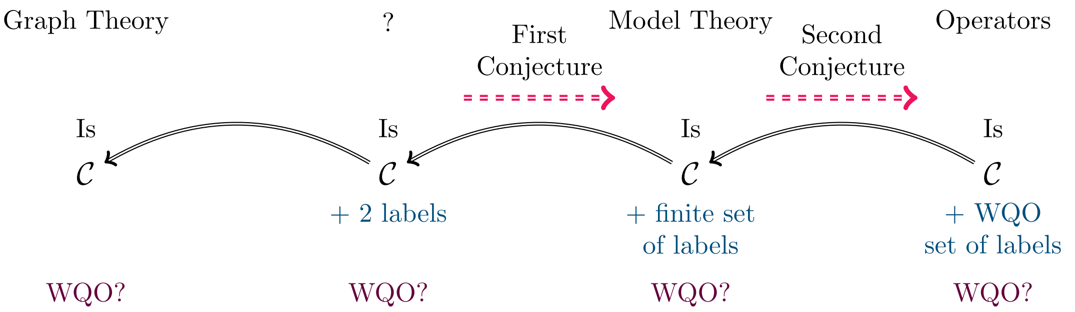
Pouzet Conjectures

Graph Theory	?	Model Theory	Operators
Is $\mathcal{C}$ WQO?	Is $\mathcal{C}$ + 2 labels WQO?	Is $\mathcal{C}$ + finite set of labels WQO?	Is $\mathcal{C}$ + WQO set of labels WQO?

Pouzet Conjectures

Graph Theory	?	Model Theory	Operators
Is $\mathcal{C}$	Is $\mathcal{C}$	Is $\mathcal{C}$	Is $\mathcal{C}$
	+ 2 labels	+ finite set of labels	+ WQO set of labels
WQO?	WQO?	WQO?	WQO?

Pouzet Conjectures



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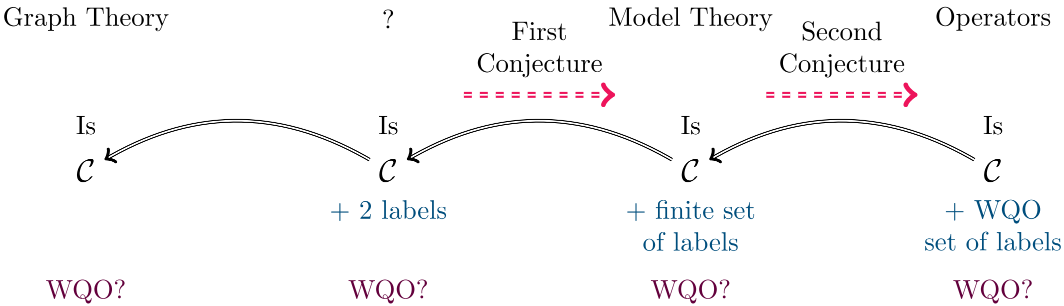

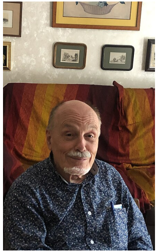

First Pouzet Conjecture: $2 = \omega$ [Pou72]

Second Pouzet Conjecture: $\omega = \mathsf{WQO}$

Schmitz's Conjecture: *finite paths* are the only obstruction to 2-WQO

Finite Paths

Finite Paths are WQO but not 2-WQO

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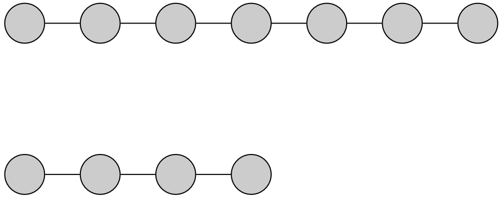

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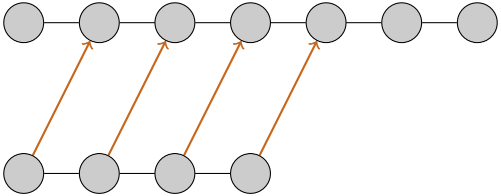

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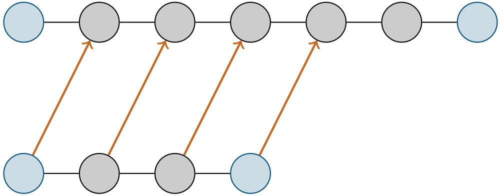

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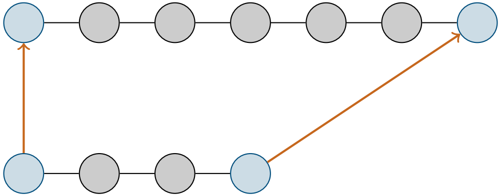
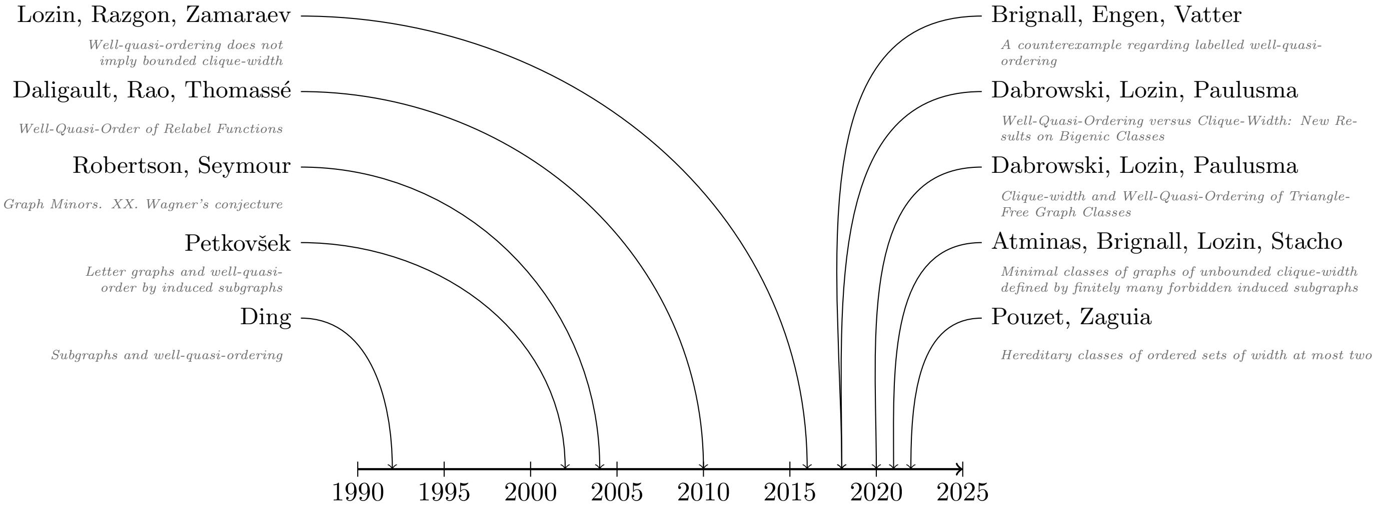


State of the Art

Well Quasi Ordering Graph Classes in 2024



Usual Proof Method

From Graphs To Trees

$$(\mathsf{Trees}, \leq) \xrightarrow{\text{monotone, surjective}} (\mathcal{C}, \subseteq_i)$$

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Nested Higman?

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Typical Example

$T \in \mathsf{Trees}(\Sigma, \mathbb{P}(\Sigma \times \Sigma)) \longrightarrow G \in \mathcal{C}$

m-partite cographs

Σ finite with $m = |\Sigma|$

order = Kruskal

$V(G) = \text{leaves of } T$

$(x, y) \in E(G) \iff (c(x), c(y)) \in \mathsf{lca}(x, y)$

Labelled
Well Quasi Ordered Classes
of
Bounded (Linear) Clique-Width

an automata based approach

Bounded Clique Width

A Logical Approach to Graph Decompositions

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$$\mathsf{Trees} \xrightarrow{\mathsf{MSO}\text{ definable, surjective}} \mathcal{C}$$

Bounded Clique Width

A Logical Approach to Graph Decompositions

Introduced by Courcelle [Cou91; Cou94; CER93] with the idea of generalizing **regular languages** to graphs.

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Bounded Clique Width

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$$\text{Trees} \xrightarrow{\text{MSO definable, surjective}} \mathcal{C}$$

$$\varphi_\Delta : \text{consider this tree?}$$
$$\varphi_{\text{dom}}(x) : \text{selects vertices of the graph}$$
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- Gives a finite representation of graph classes
- Is a well-studied parameter
- Is conjectured to be tightly related to being labelled-WQO

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Conjecture [DRT10, Conjecture 5]: $\mathcal{C}$ is hereditary and 2-WQO $\implies$ $\mathcal{C}$ has bounded clique-width

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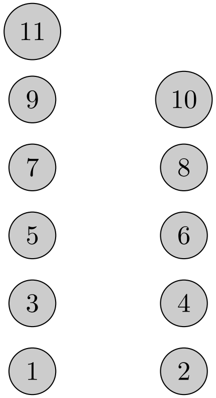
To simplify *everything* consider linear trees, i.e., **finite words**

Bounded **Linear** Clique-Width

m	o	n	t	p	e	l	l	i	e	r
1	2	3	4	5	6	7	8	9	10	11



$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

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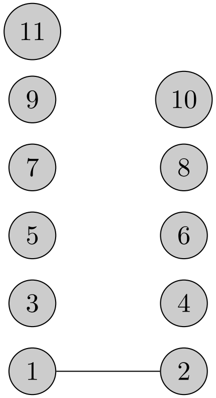

$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(1, 2) \vee \varphi(2, 1) = \top$$

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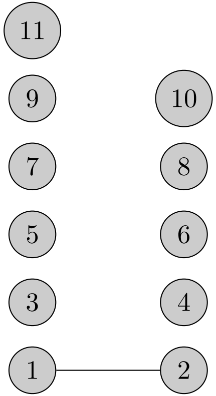
m o n t p e l l i e r

1 2 3 4 5 6 7 8 9 10 11

x y



$$\varphi(x, y) = \text{isOdd}(x) \land \text{isEven}(y)$$

$$\varphi(1, 3) \lor \varphi(3, 1) = \bot$$

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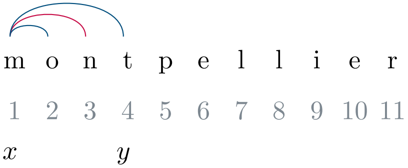
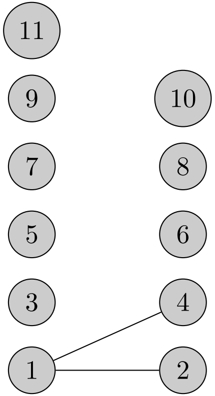
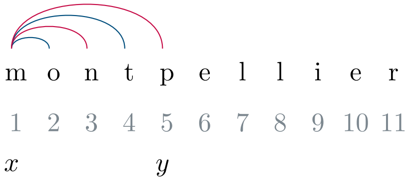


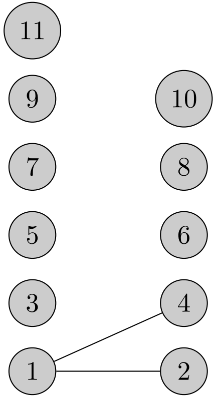


$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(1, 4) \vee \varphi(4, 1) = \top$$

Bounded **Linear** Clique-Width





$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(1, 5) \vee \varphi(5, 1) = \bot$$

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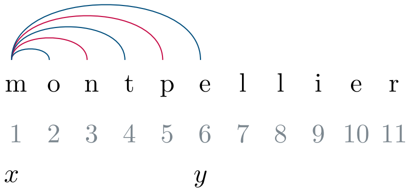
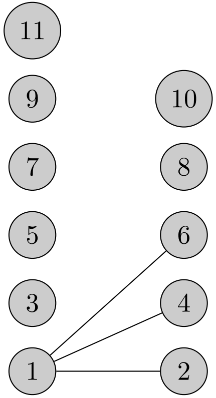
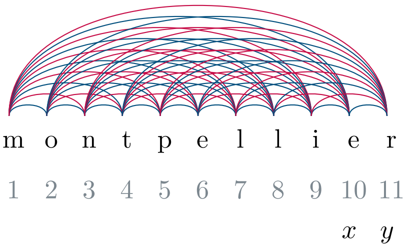


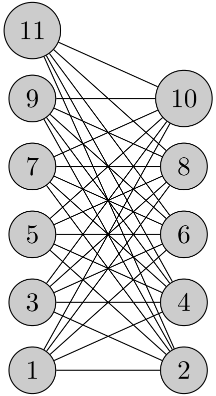


$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(1, 6) \vee \varphi(6, 1) = \top$$

Bounded **Linear** Clique-Width





$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(10, 11) \vee \varphi(11, 10) = \top$$

The Automata Toolbox

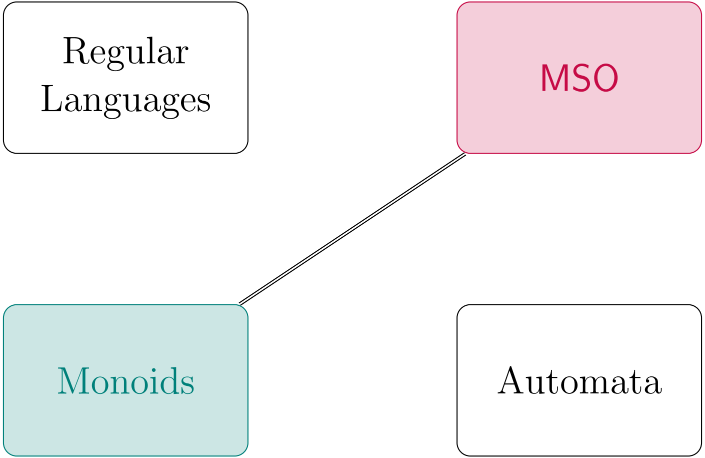
Regular Languages

MSO

Monoids

Automata

The Automata Toolbox



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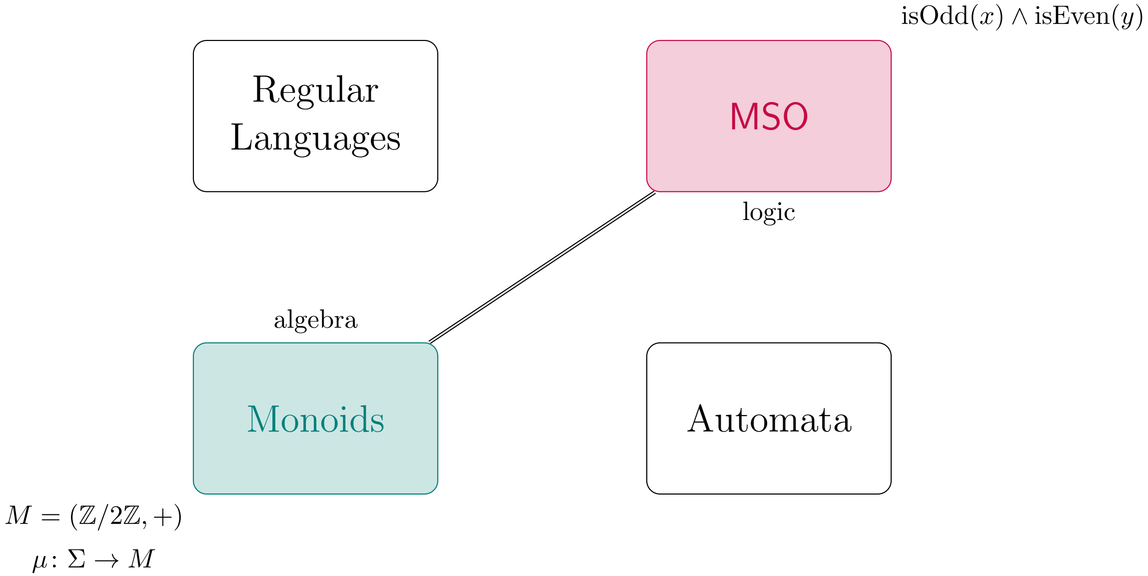

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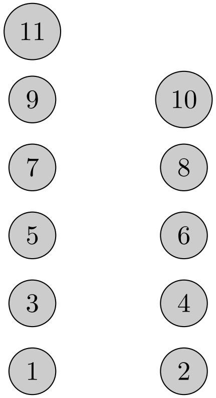

m o n t p e l l i e r

1 2 3 4 5 6 7 8 9 10 11

$M = (\mathbb{Z}/2\mathbb{Z}, +), \mu(x) = 1$

$P \subseteq M^3 = \{(x, 0, y) \mid (x, y) \in \mathbb{Z}/2\mathbb{Z}\}$

Bounded **Linear** Clique-Width

m o n t p e l l i e r

1 2 3 4 5 6 7 8 9 10 11

$\vdash$ — x - y —————— $\dashv$

0 0 1

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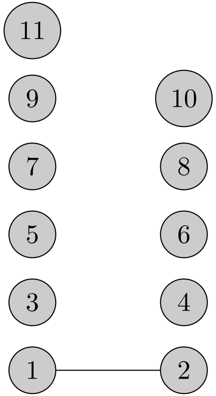


$(\mu(w[1:1]), \mu(w[1:2]), \mu(w[2:|w|])$

$= (0, 0, 1)$

$\in P?\ \top$

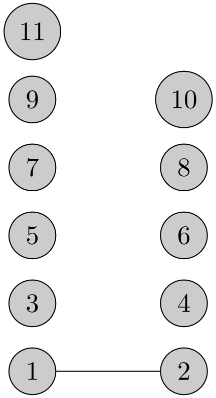
Bounded **Linear** Clique-Width

m o n t p e l l i e r

1 2 3 4 5 6 7 8 9 10 11

$\vdash$ — x — y ———— $\dashv$

0 1 0



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$$(\mu(w[1:1]), \mu(w[1:3]), \mu(w[3:|w|])$$

$$= (0, 1, 0)$$

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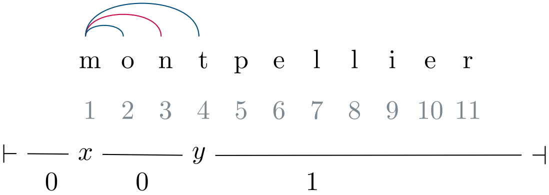

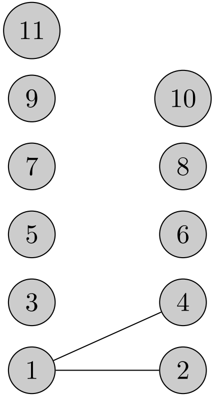

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$$(\mu(w[1:1]), \mu(w[1:4]), \mu(w[4:|w|])$$

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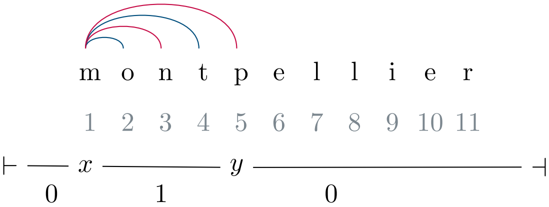

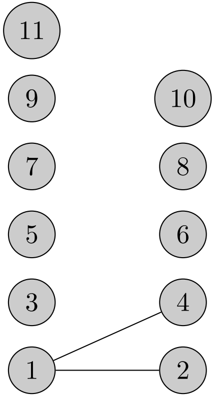

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$$(\mu(w[1:1]), \mu(w[1:5]), \mu(w[5:|w|])$$

$$= (0, 1, 0)$$

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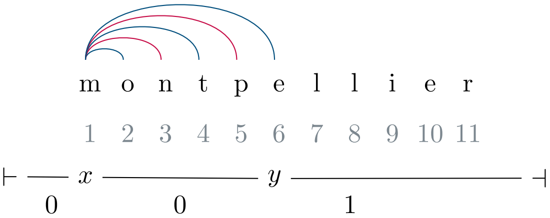

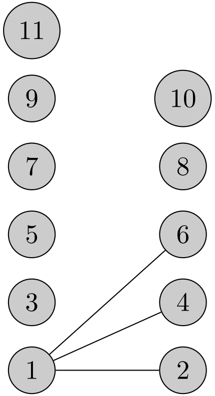

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$$(\mu(w[1:1]), \mu(w[1:6]), \mu(w[6:|w|])$$

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$$\in P?\ \top$$

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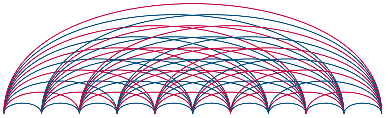

m o n t p e l l i e r

1 2 3 4 5 6 7 8 9 10 11

⊢ ———— x - y —— ⊣

1 0

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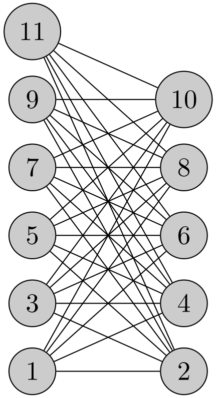

$$(\mu(w[1:10]), \mu(w[10:11]), \mu(w[11:|w|])$$

$$= (1, 0, 0)$$

$$\in P? \ \top$$

Simon's Factorisation(s)

A Ramsey-like fundamental theorem of semigroup theory [Sim90].

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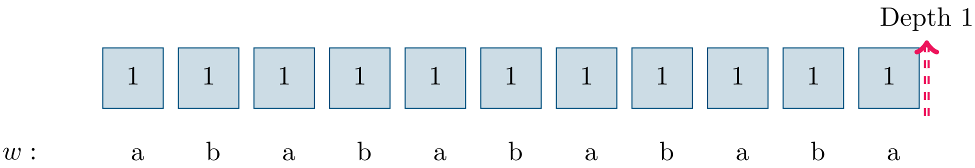

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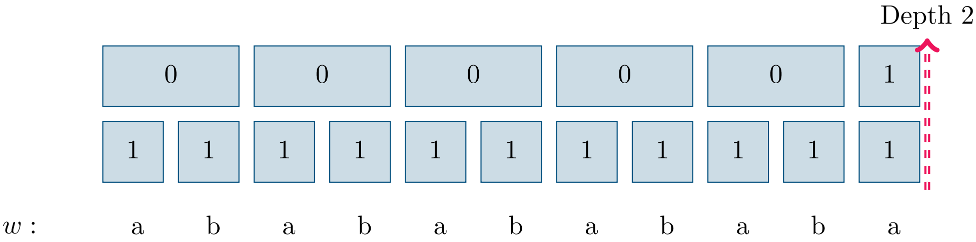

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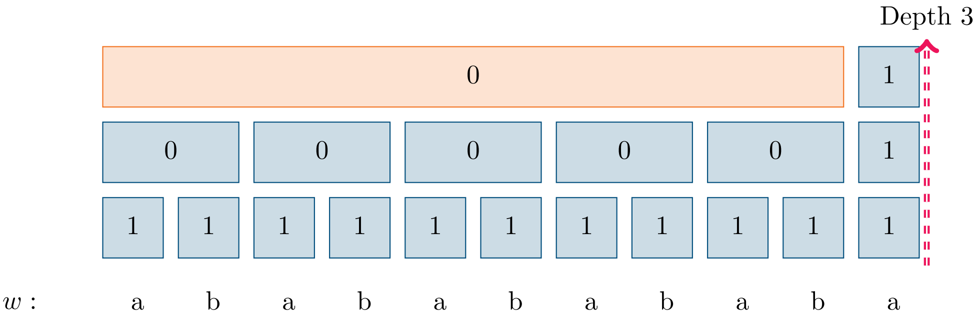

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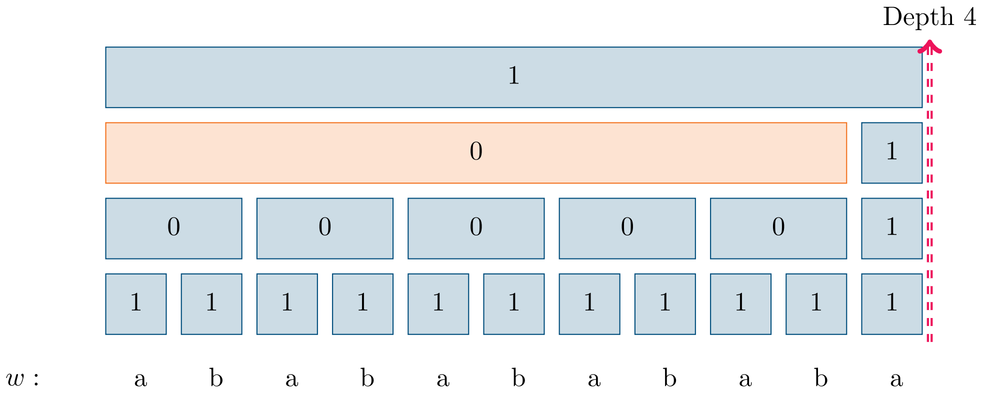

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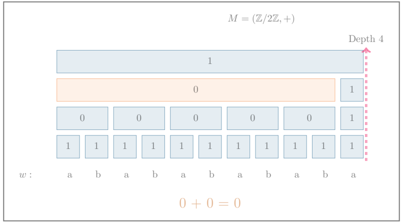

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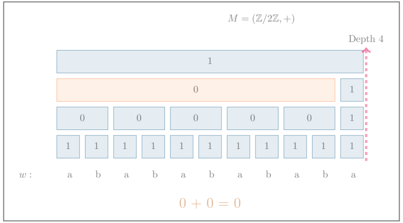

Theorem (Simon [Sim90]): for every word w and monoid M, there exists a factorisation of depth $f(|M|)$ independent of w.

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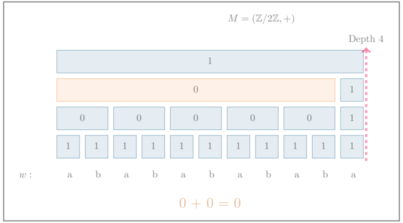

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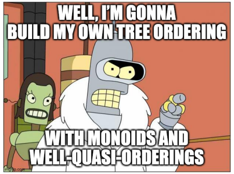

Back to the trees!

Take Away Food For Thought

If there was only one thing to remember from this talk ...

Good Orders On Factorisations $\simeq$ Well-Quasi-Ordered classes of graphs

Take Away Food For Thought

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Nested Higman Over Simon's	$\simeq$	Well-Quasi-Ordered classes of graphs

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Gap-Embedding [DT03]	$\simeq$	Well-Quasi-Ordered classes of graphs

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Gap-Embedding [DT03] $\simeq$ Well-Quasi-Ordered classes of graphs

Logic + WQO = $\heartsuit$

The Main Issue

Inserting idempotents does not affect **far away** positions...

... But does affect **close** positions!

$$M = (\{0, 1, +\infty\}, +),\ \mu(\varepsilon) = 0,\ \mu(a) = \mu(b) = 1$$

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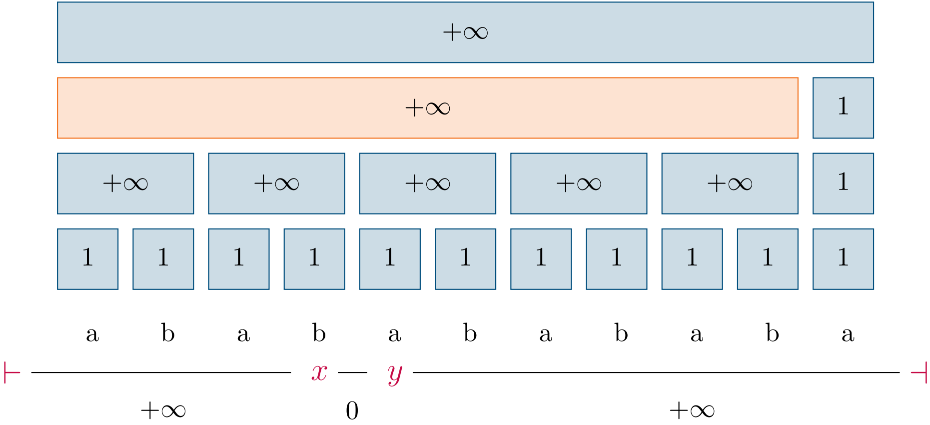

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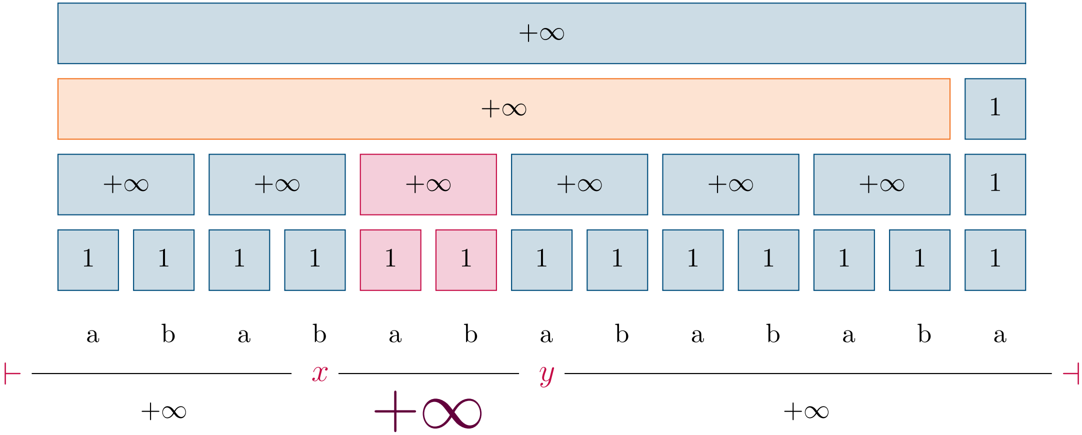

$M = (\{0, 1, +\infty\}, +),\ \mu(\varepsilon) = 0,\ \mu(a) = \mu(b) = 1$

Conclusion / Ask Me Anything

Theorem 1: Given an MSO transduction, one can **decide** if its *image* is labelled-WQO, and **compute** a k such that k-wqo $\iff$ labelled-wqo.

Theorem 2: For every class $\mathcal{C}$ of bounded *linear* clique-width, labelled-wqo $\iff$ wqo-operator.

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Remark 3: Previous works of [DRT10] can be explained in this framework, using a nice monoid variety [Kun05].

Remark 4: No *minimal bad sequence argument* [Nas65], just the *Gap-Embedding* of [DT03].

Conclusion / Ask Me Anything

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Remark 3: Previous works of [DRT10] can be explained in this framework, using a nice monoid variety [Kun05].

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Conclusion / Ask Me Anything

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In the near future:
Clique-Width (almost written up, using [Lop23]),
Schmitz's Conjecture (almost done for clique-width),
labelled-wqo implies ω-categorical (in progress).

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