Which polynomials are computed by $\mathbb{N}$-weighted automata?

and friends made along the way

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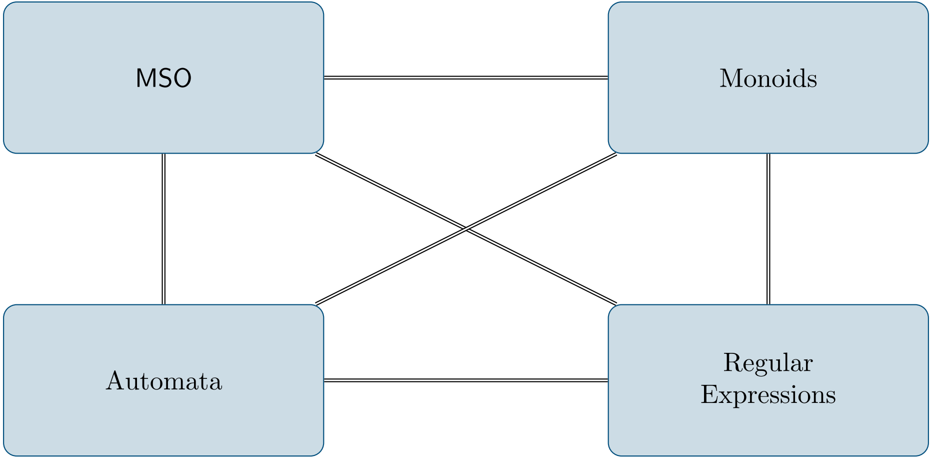
Automata Theory

Seminal results from [Büc60; Tra66; Kle56; Elg61]



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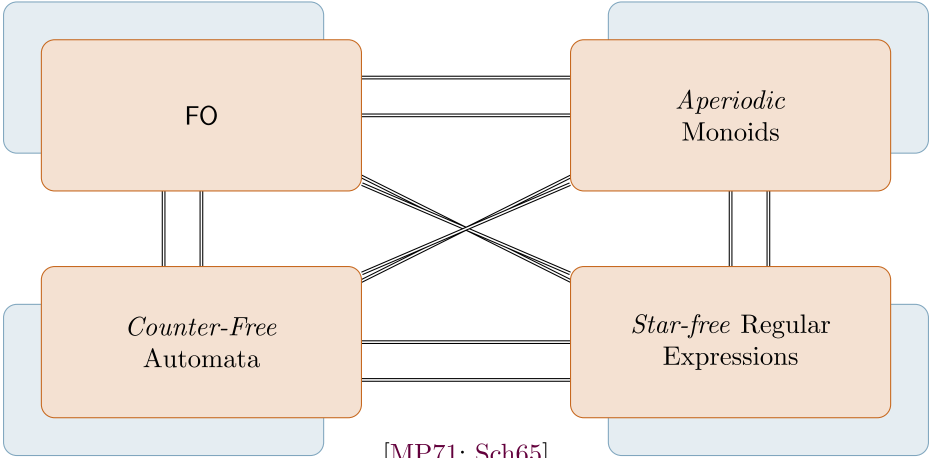

[MP71; Sch65]

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

Quantitative

$$\Sigma^* \to (\mathbb{A}, +, \times)$$

Beyond Automata

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Functional

$$\Sigma^* \to \Gamma^*$$

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

Quantitative

$$\Sigma^* \to (\mathbb{A}, +, \times)$$

Functional

$$\Sigma^* \to \Gamma^*$$

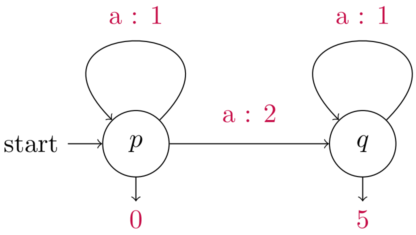
$$(\mathbb{N}, +, \times) \simeq \{1\}^*$$

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton

aaaaaa



Weighted Automata

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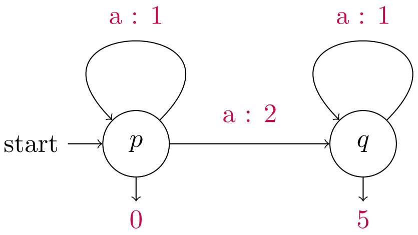
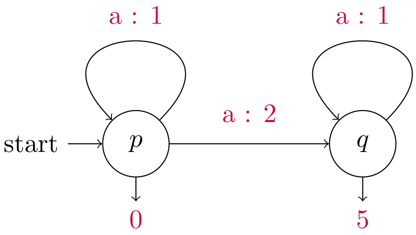


Value = 0

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

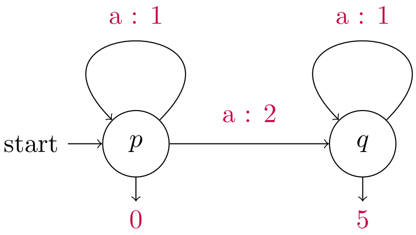
$$p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5$$

Value = 10

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

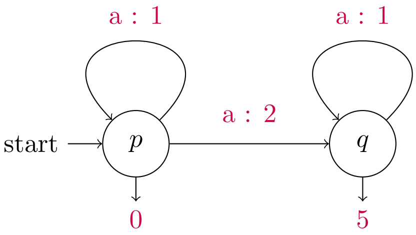
$$\mathrm{p} \xrightarrow{2\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \longrightarrow 5$$

$$+ \quad \mathrm{p} \xrightarrow{1\ \times} \mathrm{p} \xrightarrow{2\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \xrightarrow{1\ \times} \mathrm{q} \longrightarrow 5$$

$\text{Value} = 20$

Weighted Automata

as a quantitative model [Sch61]



$\mathbb{N}$-weighted automaton

aaaaaa

$$p \xrightarrow{2} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{\qquad\qquad} 5$$

$$+ \quad p \xrightarrow{1} \overset{\times}{p} \xrightarrow{2} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{\qquad\qquad} 5$$

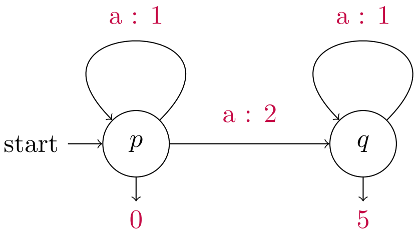
$$+ \quad p \xrightarrow{1} \overset{\times}{p} \xrightarrow{1} \overset{\times}{p} \xrightarrow{2} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{1} \overset{\times}{q} \xrightarrow{\qquad\qquad} 5$$

Value $= 30$

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

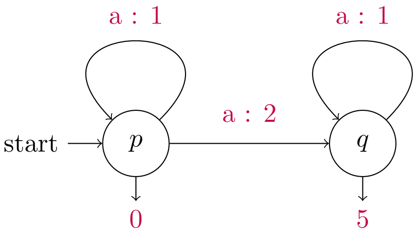
$$
\begin{array}{ll}
 & p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
+ & p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
+ & p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
+ & p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \longrightarrow 5
\end{array}
$$

Value $= 40$

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5$$
$$+\quad p \xrightarrow{1\ \times} p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5$$
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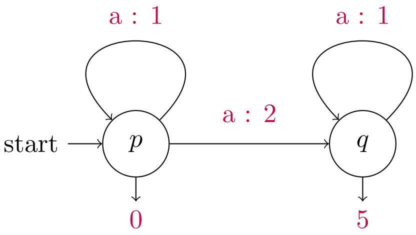
$$\text{Value} = 50$$

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$\begin{aligned}
& p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5 \\
+\ & p \xrightarrow{1\ \times} p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5 \\
+\ & p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5 \\
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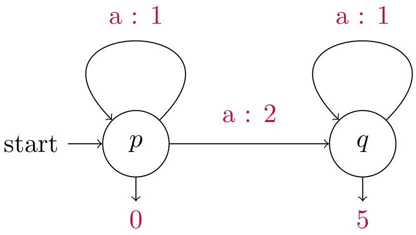
+\ & p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{2\ \times} q \longrightarrow 5 \\
+\ & p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \longrightarrow 0
\end{aligned}$$

Value $= 50$

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$
\begin{array}{ll}
 & p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5 \\
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+ & p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \xrightarrow{1\ \times} p \longrightarrow 0
\end{array}
$$

Value $= 50$

$$f(w) \mapsto 10(|w| - 1)$$

Polyregular Functions

One (among many) way to define functions [EM02; Boj18]

$$\Sigma^* \longrightarrow \Gamma^*$$

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$$(\varphi_a)_{a \in \Gamma} \in \mathsf{MSO}^\Gamma$$

$$\varphi_\leq \in \mathsf{MSO}$$

Polyregular Functions

One (among many) way to define functions [EM02; Boj18]

$$\Sigma^* \xrightarrow{\mathsf{MSO}} \Gamma^*$$

$$(\varphi_a)_{a \in \Gamma} \in \mathsf{MSO}^{\Gamma}$$

$$\varphi_{\leq} \in \mathsf{MSO}$$

One can also define FO (aperiodic) functions!

Open Problem: decide if a polyregular function is *aperiodic*.

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

$\varphi_{\leq}$ is unused

$\varphi_a(x_1, \ldots, x_n) \simeq$ counting!

ℕ-polyregular function

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$\varphi_a(x_1, \ldots, x_n) \simeq$ counting!

$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function

$\sum_{i=1}^{n} \alpha_i \# \varphi_i$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

Polynomial Growth $\wedge$ $\mathbb{N}$-weighted automaton $\qquad w \mapsto 2^{|w|}|w|$

$\Updownarrow$

$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function $\qquad \sum_{i=1}^{n} \alpha_i \# \varphi_i$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

$\exists k \in \mathbb{N}. f(w) = \mathcal{O}(|w|^k)$ $\bigwedge$ $\mathbb{N}$-weighted automaton $w \mapsto 2^{|w|}|w|$

$\Updownarrow$

$w \mapsto |w|^3 + 5|w|$
$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function

$\sum_{i=1}^{n} \alpha_i \# \varphi_i$

Now, what is the problem?

Deciding Aperiodicity is Hard

Even in the unary case!

$$\Sigma^* \longrightarrow \Gamma^* \; ?$$

Deciding Aperiodicity is Hard

Even in the unary case!

$$\begin{array}{ccc} \Sigma^* & \longrightarrow & \Gamma^* \; ? \\ & \searrow & \rotatebox{90}{$\subseteq$} \\ & & \mathbb{N} \; ? \end{array}$$

Deciding Aperiodicity is Hard

Even in the unary case!

$$
\begin{tikzcd}
\Sigma^* \arrow[rr] \arrow[rrdd] \arrow[rrddd] & & \Gamma^* \ ? \\
& & \rotatebox{90}{$\subseteq$} \\
& & \mathbb{N} \ ? \\
& & \mathbb{Z} \ \text{[CDL23]}
\end{tikzcd}
$$

Deciding Aperiodicity is Hard

Even in the unary case!

$$\begin{tikzcd}
\Sigma^* \arrow[rr] \arrow[rrd] \arrow[rrdd] & & \Gamma^* \; ? \\
& & \mathbb{N} \; ? \arrow[u, phantom, "\cup\!|"] \\
& & \mathbb{Z} \; [\textbf{CDL23}] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

$\mathbb{Z}$-weighted automata

All protagonists

With open problems from [Dou23; Kar77]

$$
\begin{tikzcd}
\mathbb{Z}\mathsf{SF} \arrow[r] & \mathbb{Z}\mathsf{Poly} \arrow[r] & \mathbb{Z}\mathsf{WA} \\
\mathbb{N}\mathsf{SF} \arrow[u, dashed] \arrow[r, dashed] \arrow[d] & \mathbb{N}\mathsf{Poly} \arrow[u, dashed] \arrow[r] \arrow[d] & \mathbb{N}\mathsf{WA} \arrow[u, dashed] \\
\mathsf{SF} \arrow[r, dashed] & \mathsf{Poly} &
\end{tikzcd}
$$

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$$\begin{tikzcd}
\mathbb{Z}\mathsf{SF} \arrow[r] & \mathbb{Z}\mathsf{Poly} \arrow[r] & \mathbb{Z}\mathsf{WA} \\
\mathbb{N}\mathsf{SF} \arrow[u, dashed] \arrow[r, dashed] \arrow[d] & \mathbb{N}\mathsf{Poly} \arrow[u, dashed] \arrow[r] \arrow[d] & \mathbb{N}\mathsf{WA} \arrow[u, dashed] \\
\mathsf{SF} \arrow[r, dashed] & \mathsf{Poly} &
\end{tikzcd}$$

All protagonists

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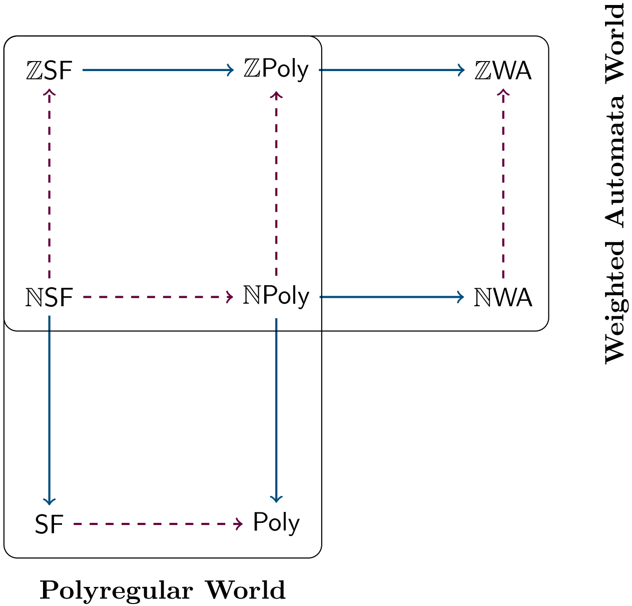
$$\begin{tikzcd}
\mathbb{Z}\mathsf{SF} \arrow[r] & \mathbb{Z}\mathsf{Poly} \arrow[r] & \mathbb{Z}\mathsf{WA} \\
\mathbb{N}\mathsf{SF} \arrow[u, dashed] \arrow[r, dashed] \arrow[d] & \mathbb{N}\mathsf{Poly} \arrow[u, dashed] \arrow[r] \arrow[d] & \mathbb{N}\mathsf{WA} \arrow[u, dashed] \\
\mathsf{SF} \arrow[r, dashed] & \mathsf{Poly} &
\end{tikzcd}$$

Polyregular World

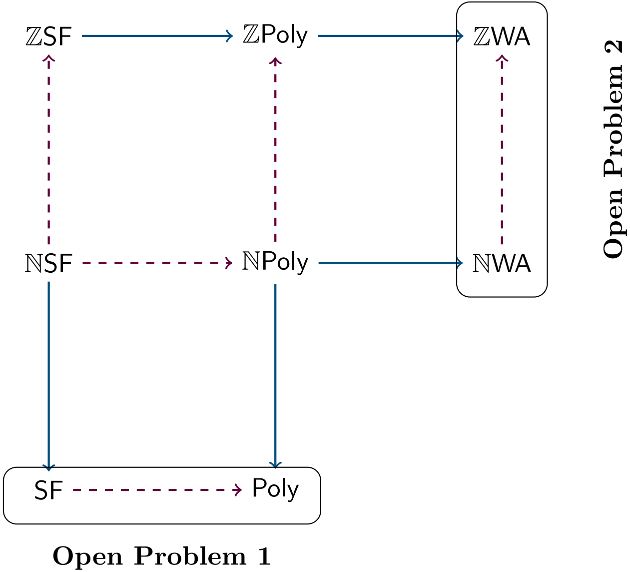
All protagonists

With open problems from [Dou23; Kar77]



All protagonists

With open problems from [Dou23; Kar77]



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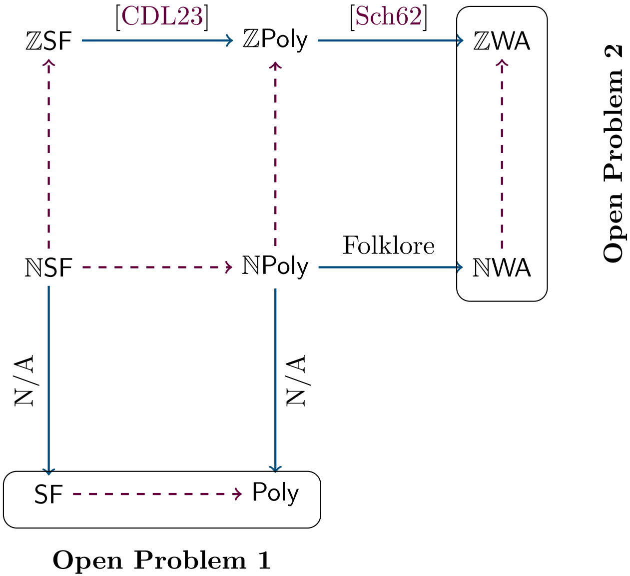
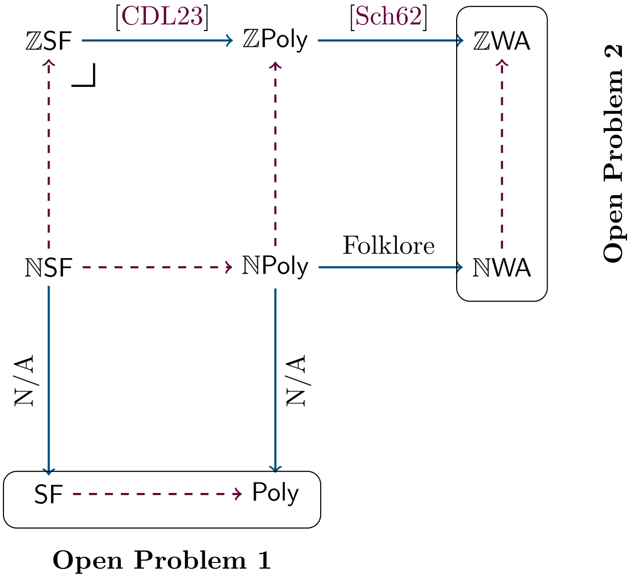


All protagonists

With open problems from [Dou23; Kar77]

$$\mathbb{Z}\mathsf{SF} \cap \mathbb{N}\mathsf{Poly} = \mathbb{Z}\mathsf{SF}$$



Simplify further: commutative input

Commutative Input?

$$\Sigma^* \xrightarrow{f} \mathbb{Z}$$

Commutative Input?

$$\begin{tikzcd} \Sigma^* \arrow[loop above, "\sigma"] \arrow[rrr, "f"] & & & \mathbb{Z} \end{tikzcd}$$

$$f(abab) = f(aabb)$$

Commutative Input?

$$\begin{tikzcd}
\Sigma^* \arrow[rr, "f"] \arrow[d, bend left] & & \mathbb{Z} \\
\mathbb{N}^\Sigma \arrow[u, bend left] \arrow[urr] & &
\end{tikzcd}$$

$$f(w) = |w|_a$$

$$f(w) = (|w|_a - |w|_b)^2$$

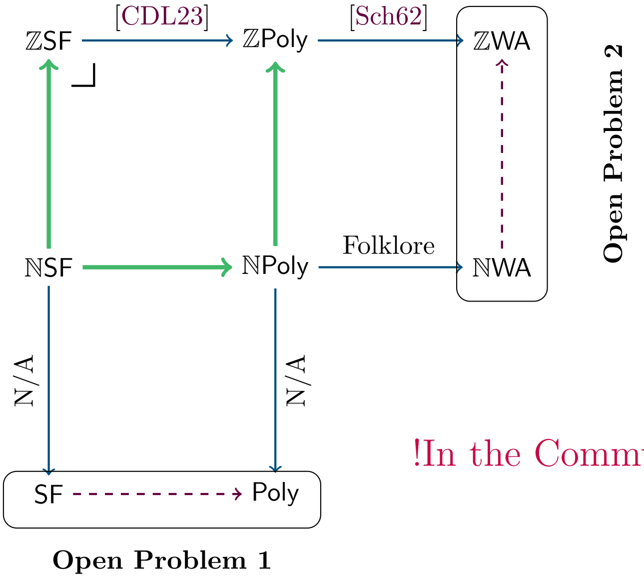
$$f(w) = |w|_a^2 - 2|w|_a|w|_b + |w|_b^2$$

In particular: polynomials in $\mathbb{N}[\vec{X}]$ and $\mathbb{Z}[\vec{X}]$

All protagonists

With open problems from [Dou23; Kar77]

$$\mathbb{Z}\mathsf{SF} \cap \mathbb{N}\mathsf{Poly} = \mathbb{Z}\mathsf{SF}$$



!In the Commutative Setting!

An easy case

$\mathbb{Z}$-polynomials are all in $\mathbb{Z}$WA... and more!

$\mathbb{Z}\mathsf{WA}$

$\cup\!|$

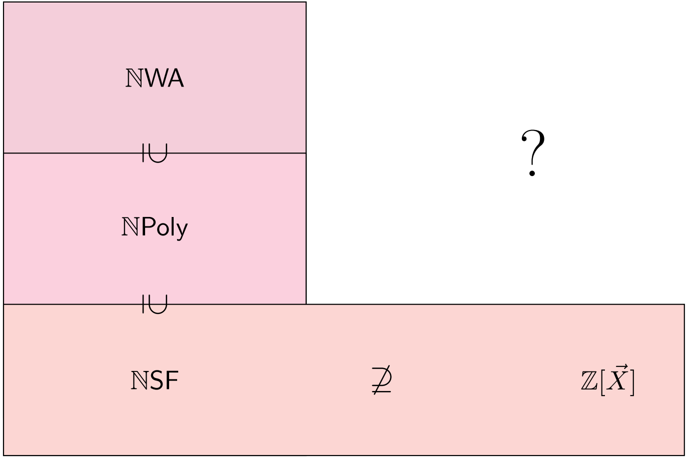
$\mathbb{Z}\mathsf{Poly}$

$\cup\!|$

$\mathbb{Z}\mathsf{SF} \supseteq \mathbb{Z}[\vec{X}]$

An easy case

$\mathbb{Z}$-polynomials are all in $\mathbb{Z}$WA... and more!



An easy case

$\mathbb{Z}$-polynomials are all in $\mathbb{Z}$WA... and more!

$$Q(X,Y) = X^2 - 2X + 1$$
$$P(X,Y) = (X-Y)^2$$

$\mathbb{N}$WA

$\supseteq$

$\mathbb{N}$Poly

$\supseteq$

$\mathbb{N}$SF $\quad \not\supseteq \quad \mathbb{Z}[\vec{X}]$

?

An easy case

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$\mathbb{N}$WA

$\cup\!|$

$\mathbb{N}$Poly

$\cup\!|$

$\mathbb{N}$SF $\quad \not\supseteq \quad$ $\mathbb{Z}[\vec{X}]$

?

Theorem [Boj18]: the pre-image of a finite set is *regular*

$$P^{-1}(\{0\}) = \{w \mid |w|_a = |w|_b\}$$

A previous statement

of Karhumäki [Kar77].

$$P = P_{\max} + P_{\text{rest}}$$

$$P_{\max} \in \mathbb{N} \qquad \forall \vec{x} \in \mathbb{N}^k, P(\vec{x}) \geq 0$$

$$\Updownarrow$$

$$P \in \mathbb{N}\mathsf{WA}$$

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$$P = P_{\max} + P_{\text{rest}}$$

$P_{\max} \in \mathbb{N}$ $\quad\quad$ $\forall \vec{x} \in \mathbb{N}^k, P(\vec{x}) \geq 0$

Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

$$\Updownarrow$$

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$$\Updownarrow$$

Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

Counterexample: $P = Z(X+Y)^2 + 2(X-Y)^2$.

$$P \in \mathbb{N}\mathsf{WA}$$

Corrected Theorem

Theorem: For all $P \in \mathbb{Z}[\vec{X}]$, there exists a computable $K \in \mathbb{N}$ such that
$P \in \mathbb{N}\mathsf{WA}$
$\iff$ $P \in \mathbb{N}\mathsf{Poly}$
$\iff$ $P \in \mathbb{N}\mathsf{SF}$
$\iff$ every partial evaluation of P has positive maximal coefficients
$\iff$ every partial evaluation up to K of P has positive maximal coefficients
$\iff$ every partial evaluation of P translated by K belongs to $\mathbb{N}[\vec{X}]$
$\iff$ every partial evaluation *up to* K of P translated by K belongs to $\mathbb{N}[\vec{X}]$

Corrected Theorem

Theorem: For all $P \in \mathbb{Z}[\vec{X}]$, there exists a computable $K \in \mathbb{N}$ such that
$P \in \mathbb{N}\mathsf{WA}$
$\iff$ $P \in \mathbb{N}\mathsf{Poly}$
$\iff$ $P \in \mathbb{N}\mathsf{SF}$
$\iff$ every partial evaluation of P has positive maximal coefficients
$\iff$ every partial evaluation up to K of P has positive maximal coefficients
$\iff$ every partial evaluation of P translated by K belongs to $\mathbb{N}[\vec{X}]$
$\iff$ every partial evaluation *up to* K of P translated by K belongs to $\mathbb{N}[\vec{X}]$

Remark: with at most two indeterminate, it is the same as [Kar77].

A short break for our sponsors: polynomial pumping arguments

Polynomial Pumping

Also known as: forcing commutativity

$$\Sigma^* \xrightarrow{\quad f \quad} \mathbb{Z}$$

Polynomial Pumping

Also known as: forcing commutativity

$$p(x_1, \ldots, x_k) = v_0 \prod_{i=1}^{n} (u_i)^{x_i} v_i$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f"] & \mathbb{Z} \\ \mathbb{N}^k \arrow[u, "p"] \arrow[ur, dashed] & \end{tikzcd}$$

Binomial Coefficients

as polynomials

$$\textbf{Binomial: } \binom{X-p}{k} = \frac{1}{k!}(X-p)(X-p-1)\cdots(X-p-k+1)$$

Binomial Coefficients

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Natural Binomial Polynomials: positive linear combinations of products of binomials

Binomial Coefficients

as polynomials

$$\textbf{Binomial: } \binom{X-p}{k} = \frac{1}{k!}(X-p)(X-p-1)\cdots(X-p-k+1)$$

Natural Binomial Polynomials: positive linear combinations of products of binomials

Theorem: $\forall f, \forall p, \exists \omega \in \mathbb{N}, \exists K \in \mathbb{N}$ such that

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{N}\mathsf{Poly}"] & \mathbb{Z} \\ (\omega\mathbb{N}_{\geq 1})^k \arrow[u, "p"] \arrow[ur, "\in \mathrm{NBP}"'] & \end{tikzcd}$$

Binomial Coefficients

as polynomials

$$\textbf{Binomial: } \binom{X-p}{k} = \frac{1}{k!}(X-p)(X-p-1)\cdots(X-p-k+1)$$

Natural Binomial Polynomials: positive linear combinations of products of binomials

Theorem: $\forall f, \forall p, \exists \omega \in \mathbb{N}, \exists K \in \mathbb{N}$ such that

$$\begin{tikzcd}
\Sigma^* \arrow[rr, "f \in \mathbb{N}\mathsf{Poly}"] & & \mathbb{Z} \\
(\omega\mathbb{N}_{\geq 1})^k \arrow[u, "p"] \arrow[urr, "\in \mathrm{NBP}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^* \arrow[rr, "f \in \mathbb{N}\mathsf{SF}"] & & \mathbb{Z} \\
(\mathbb{N}_{\geq K})^k \arrow[u, "p"] \arrow[urr, "\in \mathrm{NBP}"'] & &
\end{tikzcd}$$

Back to $\mathbb{Z}$

Let $P \in \mathbb{Q}[\vec{X}]$

$$\begin{tikzcd}
P(\mathbb{Z}^k) \subseteq \mathbb{Z} \arrow[r, Rightarrow] & P(\mathbb{N}^k) \subseteq \mathbb{Z} \arrow[r, Rightarrow, "\text{Classical result from Pólya [Pól15]}"] & P \in IBP \arrow[d, Rightarrow, "\text{Binomials are in } \mathbb{Z}\mathsf{SF}"] \\
P \in \mathbb{Z}\mathsf{WA} \arrow[u, Rightarrow] & P \in \mathbb{Z}\mathsf{Poly} \arrow[l, Rightarrow] & P \in \mathbb{Z}\mathsf{SF} \arrow[l, Rightarrow]
\end{tikzcd}$$

Integer Binomial Polynomials: integer linear combinations of products of binomials.

$$\binom{|w|}{\ell} = \#\left(\bigwedge_{1 \leq i \neq j \leq \ell} x_i \neq x_j\right)(w)$$

And now for $\mathbb{N}$

Let $P \in \mathbb{Q}[\vec{X}]$

$$
\begin{tikzcd}
\begin{array}{c} P(\mathbb{N}^k) \subseteq \mathbb{Z} \wedge \\ \alpha P \in \mathbb{N}\mathsf{WA} \end{array} \arrow[r, Rightarrow] & \begin{array}{c} P(\mathbb{N}^k) \subseteq \mathbb{Z} \wedge \\ \alpha P \in \mathbb{N}\mathsf{WA} \end{array} \arrow[r, Rightarrow, "\text{Identification of coefficients}"] & P \in SNBP \arrow[d, Rightarrow, "\text{Binomials are in } \mathbb{N}\mathsf{SF}"] \\
P \in \mathbb{N}\mathsf{WA} \arrow[u, Rightarrow] & P \in \mathbb{N}\mathsf{Poly} \arrow[l, Rightarrow] & P \in \mathbb{N}\mathsf{SF} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Strongly Natural Binomial Polynomials: Natural Binomial Polynomials, stable under partial evaluations

Tying the knot: commutative input

Commutative Input

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\omega\mathbb{N}_{\geq 1})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

Commutative Input

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\omega\mathbb{N}_{\geq 1})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

$\exists \omega \Updownarrow$ $\Updownarrow \exists K$

$f \in \mathbb{N}\mathsf{Poly}$ $f \in \mathbb{N}\mathsf{SF}$

Commutative Input

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\omega\mathbb{N}_{\geq 1})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathrm{SNBP}"'] & \end{tikzcd}$$

$$\exists\omega \Updownarrow \qquad\qquad \Updownarrow \exists K$$

$$f \in \mathbb{N}\mathsf{Poly} \qquad\qquad f \in \mathbb{N}\mathsf{SF}$$

Conjecture: true without commutative assumption

Previously, in $\mathbb{Z}$-polyregular functions

Theorems from [CDL23]

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\mathcal{O}(|w|^{d \times \deg(p)})"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathbb{Q}[\vec{X}]"'] & \end{tikzcd}$$

Previously, in $\mathbb{Z}$-polyregular functions

Theorems from [CDL23]

$$
\begin{tikzcd}
\Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\
(\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\mathcal{O}(|w|^{d \times \deg(p)})"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\
(\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathbb{Q}[\vec{X}]"'] &
\end{tikzcd}
$$

$$\exists K \Updownarrow \qquad\qquad \Updownarrow \exists K$$

$$|f(w)| = \mathcal{O}(|w|^d) \qquad\qquad f \in \mathbb{Z}\mathsf{SF}$$

Previously, in $\mathbb{Z}$-polyregular functions

Theorems from [CDL23]

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\mathcal{O}(|w|^{d \times \deg(p)})"'] & \end{tikzcd}$$

$$\begin{tikzcd} \Sigma^* \arrow[r, "f \in \mathbb{Z}\mathsf{Poly}"] & \mathbb{Z} \\ (\mathbb{N}_{\geq K})^k \arrow[u, "\forall p"] \arrow[ur, "\in \mathbb{Q}[\vec{X}]"'] & \end{tikzcd}$$

$\exists K \Updownarrow$ $\Updownarrow \exists K$

$$|f(w)| = \mathcal{O}(|w|^d) \qquad f \in \mathbb{Z}\mathsf{SF}$$

Consequence: $\mathbb{N}\mathsf{SF} = \mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF}$ in the commutative setting!

Conclusion

$$\begin{tikzcd}
\mathbb{Z}\mathsf{SF} \arrow[r, "\text{[CDL23]}"] & \mathbb{Z}\mathsf{Poly} \arrow[r, "\text{[Sch62]}"] & \mathbb{Z}\mathsf{WA} \\
\mathbb{N}\mathsf{SF} \arrow[u] \arrow[r] \arrow[d, "\text{N/A}"] & \mathbb{N}\mathsf{Poly} \arrow[u] \arrow[r, "\text{Folklore}"] \arrow[d, "\text{N/A}"] & \mathbb{N}\mathsf{WA} \arrow[u, dashed] \\
\mathsf{SF} \arrow[r, dashed] & \mathsf{Poly} &
\end{tikzcd}$$

Open Problem 2

Open Problem 1

Towards Series: define *aperiodic* $\mathbb{Z}\mathsf{WA}$ and $\mathbb{N}\mathsf{WA}$ (but not like [DG05; DG19]).

Spectrum in $\mathbb{R}_{\geq 0}$, aperiodic finite quotients, pumping arguments

Dropping Commutation: attempted a WQO approach building a canonical model [Lop24].

Using aperiodic approximations?

!In the Commutative Setting!

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