Which polynomials are computed by $\mathbb{N}$-weighted automata?

and friends made along the way

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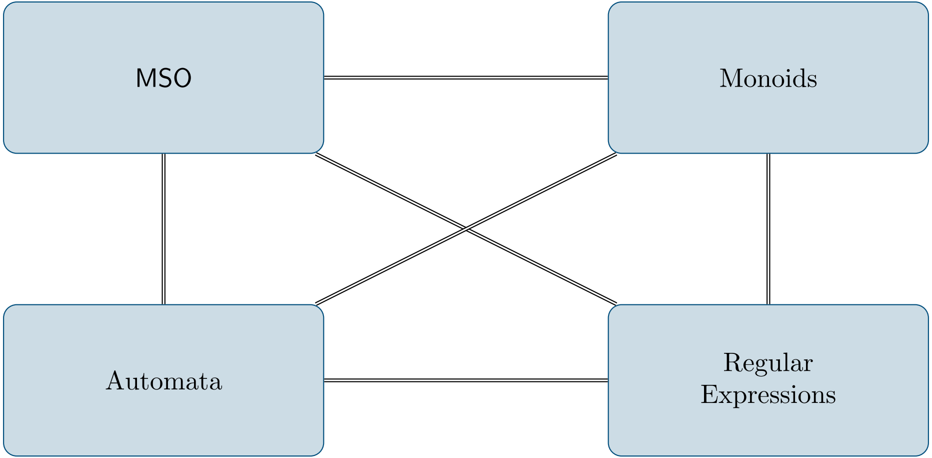
Automata Theory

Seminal results from [Büc60; Tra66; Kle56; Elg61]



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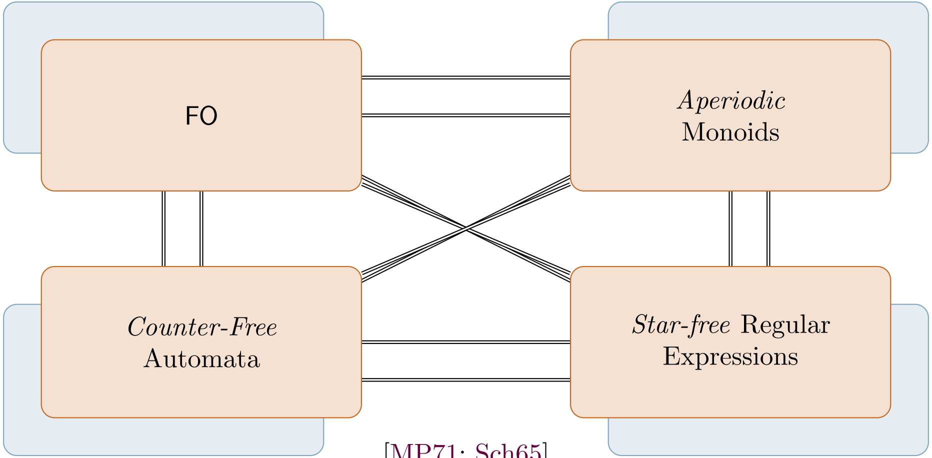

[MP71; Sch65]

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

Beyond Automata

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$$\Sigma^* \to \mathbb{B}$$

Quantitative

$$\Sigma^* \to (\mathbb{A}, +, \times)$$

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$$\Sigma^* \to (\mathbb{A}, +, \times)$$

Functional

$$\Sigma^* \to \Gamma^*$$

Beyond Automata

Extending these results...

$$\Sigma^* \to \mathbb{B}$$

Quantitative

$$\Sigma^* \to (\mathbb{A}, +, \times)$$

Functional

$$\Sigma^* \to \Gamma^*$$

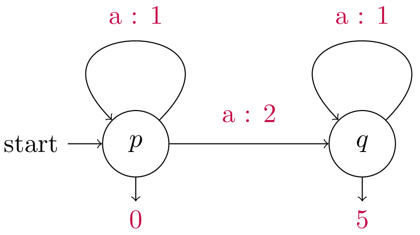
$$(\mathbb{N}, +, \times) \simeq \{1\}^*$$

Weighted Automata

as a quantitative model [Sch61]

ℕ-weighted automaton

aaaaaa



Weighted Automata

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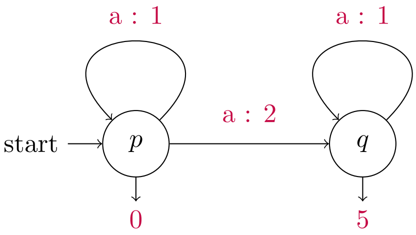
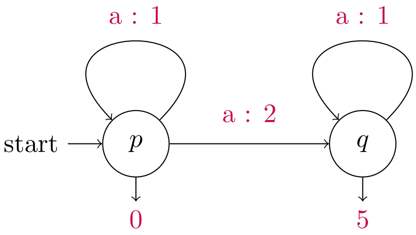


Value = 0

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



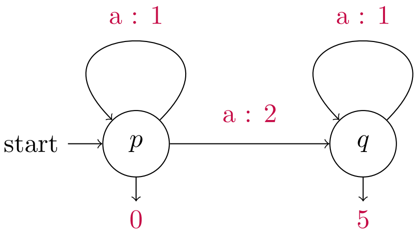
aaaaaa

$$p \xrightarrow{2\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \xrightarrow{1\ \times} q \longrightarrow 5$$

Value = 10

Weighted Automata

as a quantitative model [Sch61]



$\mathbb{N}$-weighted automaton

aaaaaa

$$p \xrightarrow{2} q \xrightarrow{1} q \xrightarrow{1} q \xrightarrow{1} q \xrightarrow{1} q \longrightarrow 5 \quad (2 \times 1 \times 1 \times 1 \times 1 \times 5)$$

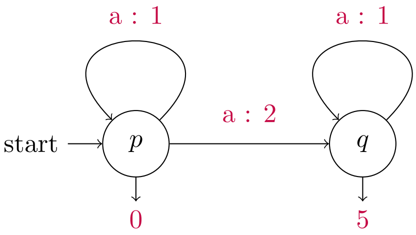
$$+ \quad p \xrightarrow{1} p \xrightarrow{2} q \xrightarrow{1} q \xrightarrow{1} q \xrightarrow{1} q \longrightarrow 5 \quad (1 \times 2 \times 1 \times 1 \times 1 \times 5)$$

Value = 20

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$p \xrightarrow{2\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \longrightarrow 5$$

$$+\quad p \xrightarrow{1\,\times} p \xrightarrow{2\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \longrightarrow 5$$

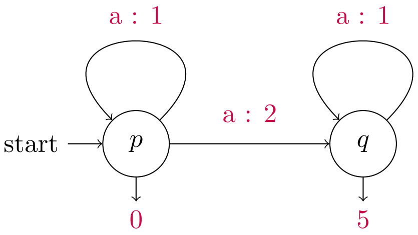
$$+\quad p \xrightarrow{1\,\times} p \xrightarrow{1\,\times} p \xrightarrow{2\,\times} q \xrightarrow{1\,\times} q \xrightarrow{1\,\times} q \longrightarrow 5$$

Value = 30

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5$$

$$+ \quad p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5$$

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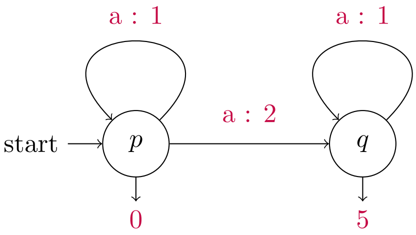
Value = 40

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

$$p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5$$
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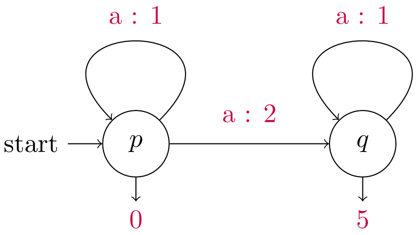
Value = 50

Weighted Automata

as a quantitative model [Sch61]



$\mathbb{N}$-weighted automaton

aaaaaa

$$\begin{aligned}
& p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
+ \; & p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
+ \; & p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{2 \times} q \xrightarrow{1 \times} q \xrightarrow{1 \times} q \longrightarrow 5 \\
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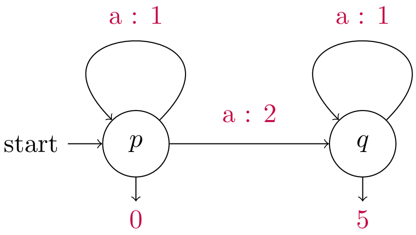
+ \; & p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \xrightarrow{1 \times} p \longrightarrow 0
\end{aligned}$$

Value = 50

Weighted Automata

as a quantitative model [Sch61]

$\mathbb{N}$-weighted automaton



aaaaaa

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Value = 50

$$f(w) \mapsto 10(|w| - 1)$$

Example Polynomials

in $\mathbb{Q}[\vec{X}]$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$$\boxed{P(X) = X^2 - 2X + 1} \qquad Q(X) = (X-1)^2$$

Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$P(X) = X^2 - 2X + 1$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$$R(X) = X(X-1)/2$$

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Example Polynomials

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$$R(X) = X(X-1)/2 \qquad S(X,Y) = -XY/3$$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$$T(X, Y) = (X - Y)^2$$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

$$T(X,Y) = (X-Y)^2$$

$$U(X,Y,Z) = Z(X+Y)^2 - 2(X-Y)^2$$

$$R(X) = X(X-1)/2$$

$$S(X,Y) = -XY/3$$

$$P(X) = X^2 - 2X + 1$$

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Example Polynomials

in $\mathbb{Q}[\vec{X}]$

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$S(X,Y) = -XY/3$

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Now, why is it interesting?

Polyregular Functions

One (among many) way to define functions [EM02; Boj18]

$$\Sigma^* \longrightarrow \Gamma^*$$

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$$\varphi_{\leq} \in \mathsf{MSO}$$

Polyregular Functions

One (among many) way to define functions [EM02; Boj18]

$$\Sigma^* \xrightarrow{\quad \mathsf{MSO} \quad} \Gamma^*$$

$$(\varphi_a)_{a \in \Gamma} \in \mathsf{MSO}^{\Gamma}$$

$$\varphi_{\leq} \in \mathsf{MSO}$$

One can also define FO (aperiodic) functions!

Open Problem: decide if a polyregular function is *aperiodic*.

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

$\varphi_{\leq}$ is unused

$\varphi_a(x_1, \ldots, x_n) \simeq$ counting!

ℕ-polyregular function

Unary Polyregular Functions

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$\varphi_a(x_1, \dots, x_n) \simeq$ counting!

$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function

$\sum_{i=1}^{n} \alpha_i \# \varphi_i$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

Polynomial Growth $\wedge$ $\mathbb{N}$-weighted automaton $\qquad w \mapsto 2^{|w|}|w|$

$\Updownarrow$

$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function $\qquad \sum_{i=1}^{n} \alpha_i \# \varphi_i$

Unary Polyregular Functions

An easier setting, and a subclass of weighted automata! [Sch62]

$\exists k \in \mathbb{N}. f(w) = \mathcal{O}(|w|^k)$ $\bigwedge$ $\mathbb{N}$-weighted automaton $w \mapsto 2^{|w|}|w|$

$\Updownarrow$

$w \mapsto |w|^3 + 5|w|$

$w \mapsto |w|_a \times |w|_b$

$\mathbb{N}$-polyregular function $\sum_{i=1}^{n} \alpha_i \# \varphi_i$

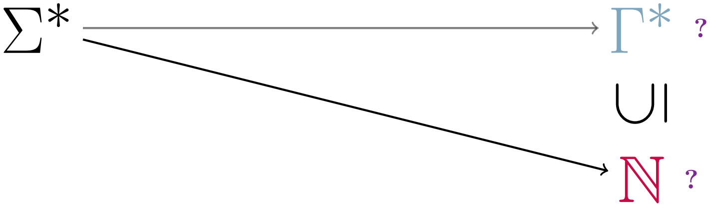
Deciding Aperiodicity is Hard

Even in the unary case!

$$\Sigma^* \longrightarrow \Gamma^* \ ?$$

Deciding Aperiodicity is Hard

Even in the unary case!



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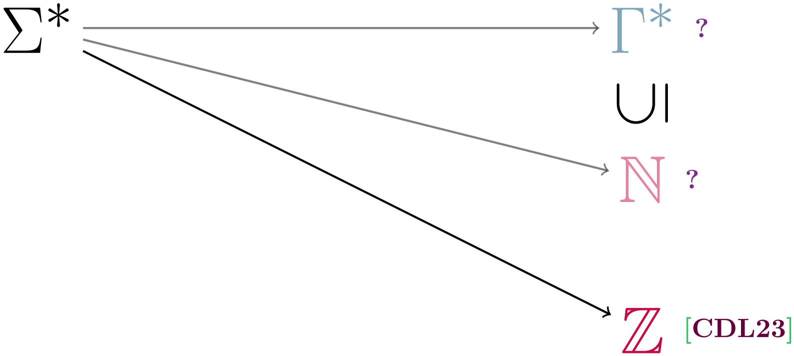

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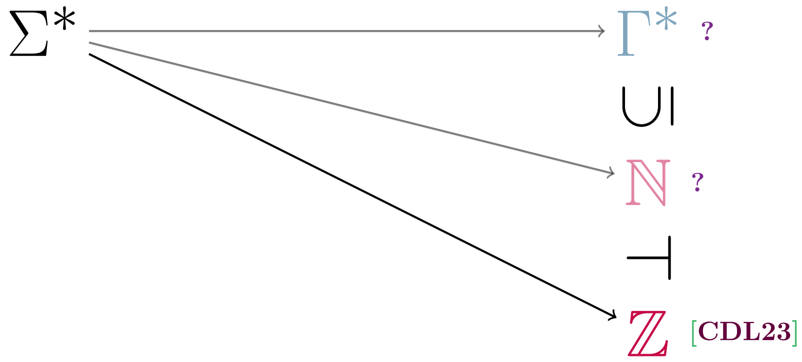
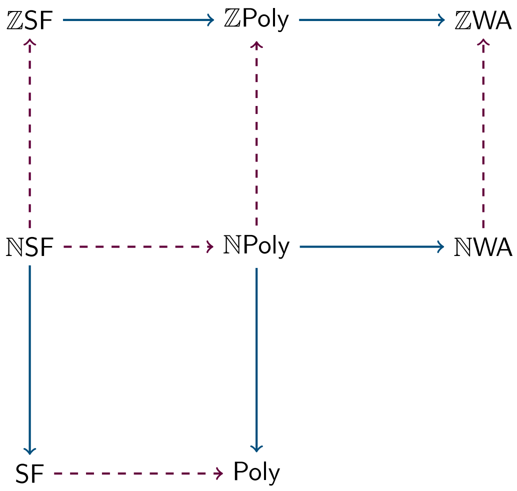


$\mathbb{Z}$-weighted automata

All protagonists

With open problems from [Dou23; Kar77]



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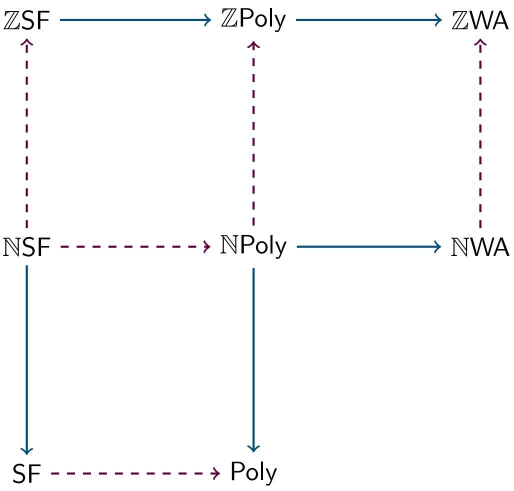

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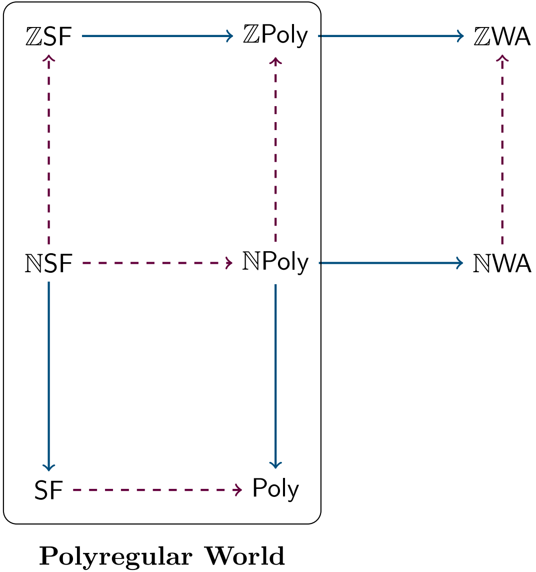

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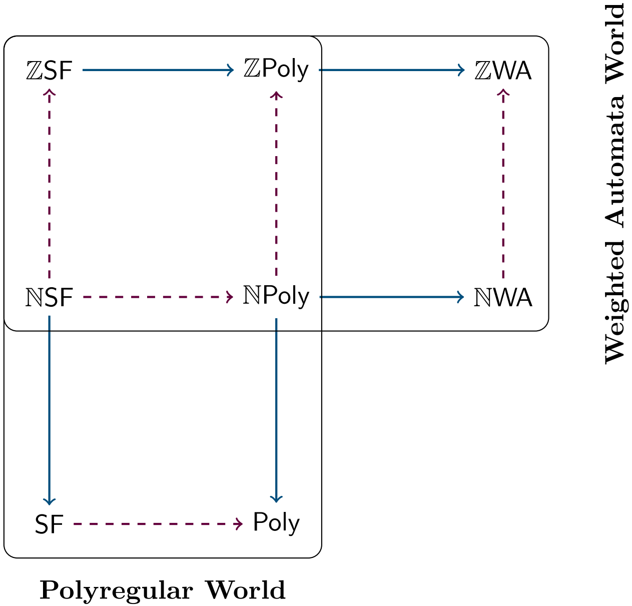

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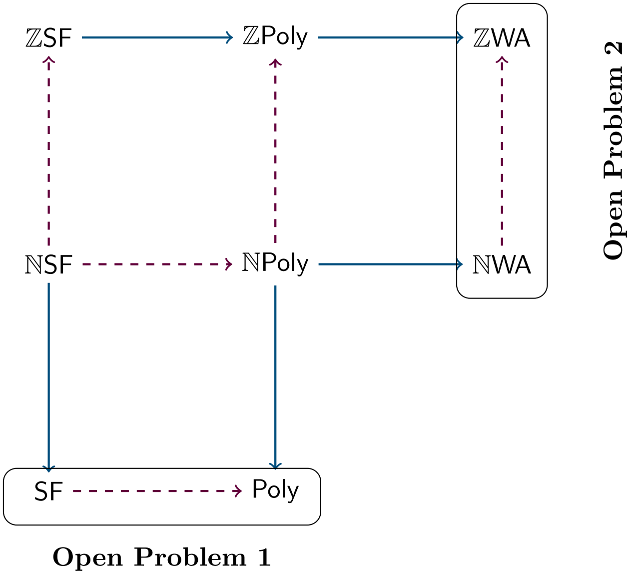

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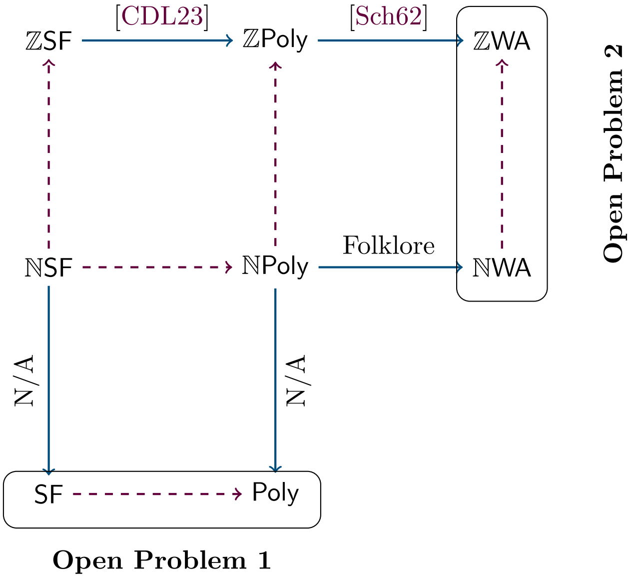
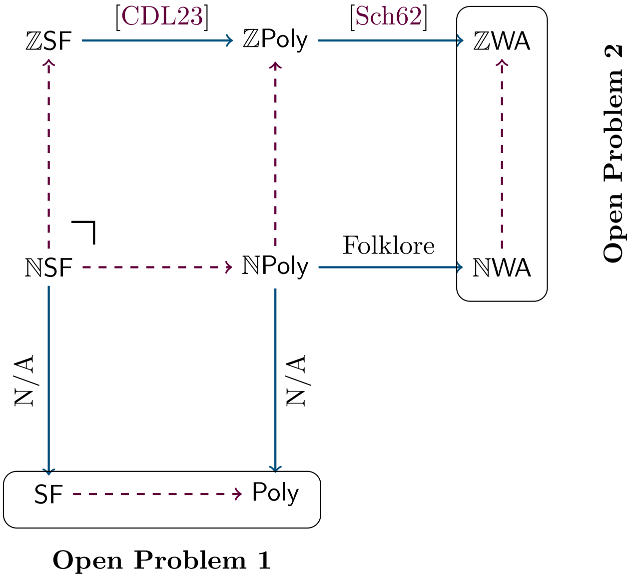


All protagonists

With open problems from [Dou23; Kar77]

$\mathbb{Z}\mathsf{SF} \cap \mathbb{N}\mathsf{Poly} = \mathbb{N}\mathsf{SF}$

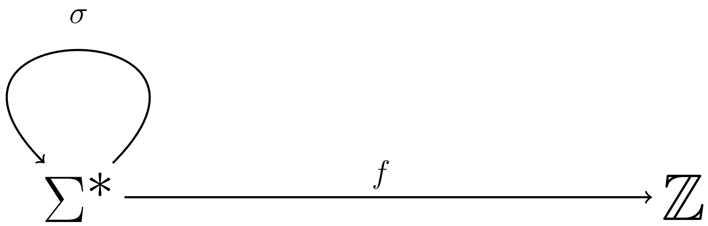


Simplify further: commutative input

Commutative Input?

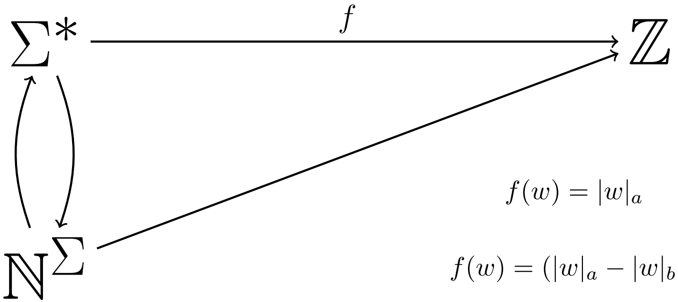
$$\Sigma^* \xrightarrow{\quad f \quad} \mathbb{Z}$$

Commutative Input?



$$f(abab) = f(aabb)$$

Commutative Input?



$$f(w) = |w|_a$$

$$f(w) = (|w|_a - |w|_b)^2$$

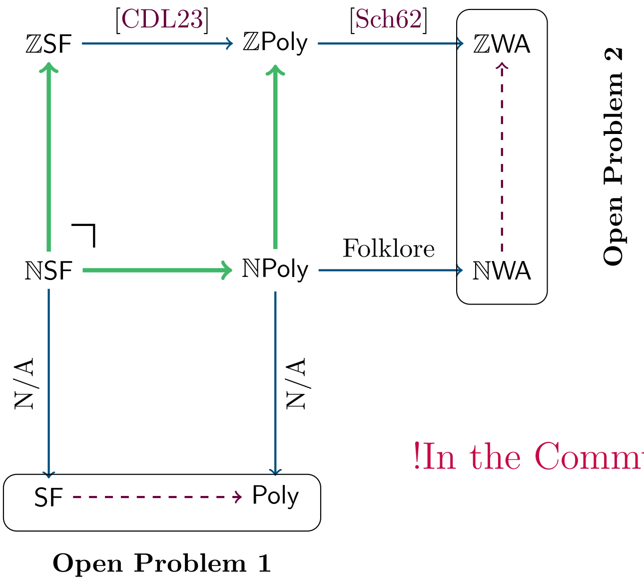
$$f(w) = |w|_a^2 - 2|w|_a|w|_b + |w|_b^2$$

In particular: polynomials in $\mathbb{N}[\vec{X}]$ and $\mathbb{Z}[\vec{X}]$

All protagonists

With open problems from [Dou23; Kar77]

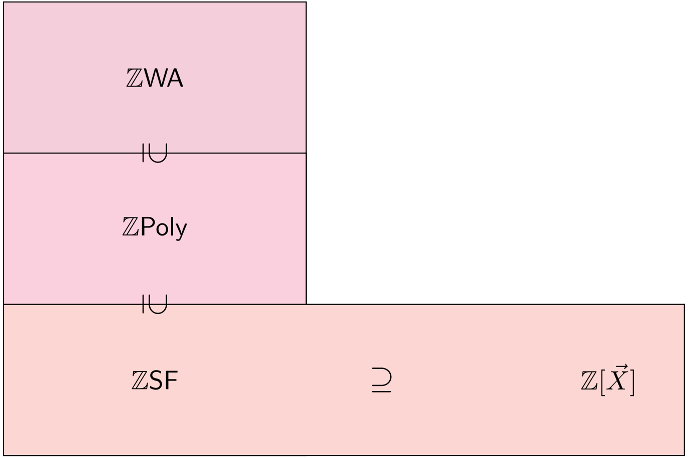
$\mathbb{Z}\mathsf{SF} \cap \mathbb{N}\mathsf{Poly} = \mathbb{N}\mathsf{SF}$



!In the Commutative Setting!

An easy case

$\mathbb{Z}$-polynomials are all in $\mathbb{Z}$WA... and more!



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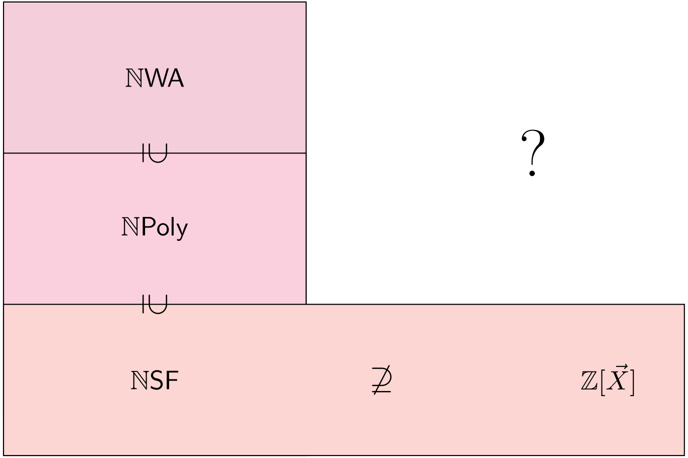

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$$Q(X,Y) = X^2 - 2X + 1$$
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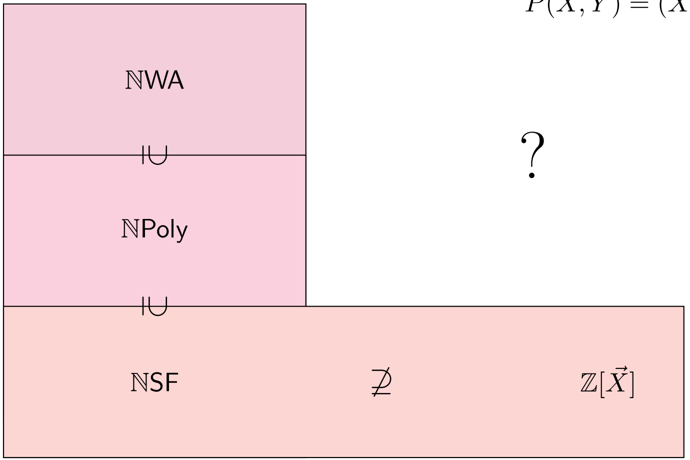

?

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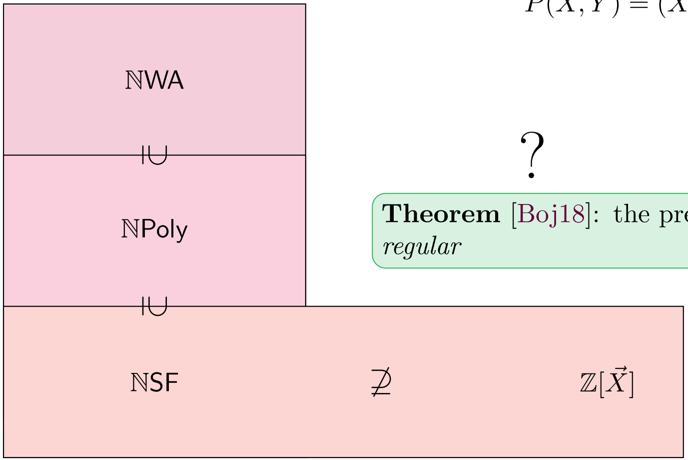

?

Theorem [Boj18]: the pre-image of a finite set is *regular*

$$P^{-1}(\{0\}) = \{w \mid |w|_a = |w|_b\}$$

A previous statement

of Karhumäki [Kar77].

$$P = P_{\max} + P_{\text{rest}}$$

$P_{\max} \in \mathbb{N}[\vec{X}]$ $\qquad$ $\forall \vec{x} \in \mathbb{N}^k, P(\vec{x}) \geq 0$

$$\Updownarrow$$

$$P \in \mathbb{N}\mathsf{WA}$$

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$$\Updownarrow$$

Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

$$P \in \mathbb{N}\mathsf{WA}$$

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Idea: $P(X_1 + K, \ldots, X_n + K) \in \mathbb{N}[\vec{X}]$ for some large enough K and *hardcode* small values.

$\Updownarrow$

Counterexample: $P = Z(X+Y)^2 + 2(X-Y)^2$.

$$P \in \mathbb{N}\mathsf{WA}$$

Corrected Theorem

Theorem: For all $P \in \mathbb{Z}[\vec{X}]$, there exists a computable $K \in \mathbb{N}$ such that
$P \in \mathbb{N}\mathsf{WA}$
$\iff$ $P \in \mathbb{N}\mathsf{Poly}$
$\iff$ $P \in \mathbb{N}\mathsf{SF}$
$\iff$ every partial evaluation of P has positive maximal coefficients
$\iff$ every partial evaluation up to K of P has positive maximal coefficients
$\iff$ every partial evaluation of P translated by K belongs to $\mathbb{N}[\vec{X}]$
$\iff$ every partial evaluation *up to* K of P translated by K belongs to $\mathbb{N}[\vec{X}]$

Corrected Theorem

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$\iff$ every partial evaluation of P translated by K belongs to $\mathbb{N}[\vec{X}]$
$\iff$ every partial evaluation *up to* K of P translated by K belongs to $\mathbb{N}[\vec{X}]$

Remark: with at most two indeterminate, it is the same as [Kar77].

A short break for our sponsors: polynomial pumping arguments

Polynomial Pumping

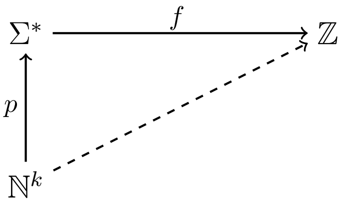
Also known as: forcing commutativity

$$\Sigma^* \xrightarrow{\quad f \quad} \mathbb{Z}$$

Polynomial Pumping

Also known as: forcing commutativity

$$p(x_1, \ldots, x_k) = v_0 \prod_{i=1}^{n} (u_i)^{x_i} v_i$$



Binomial Coefficients

as polynomials

$$\textbf{Binomial: } \binom{X-p}{k} = \frac{1}{k!}(X-p)(X-p-1)\cdots(X-p-k+1)$$

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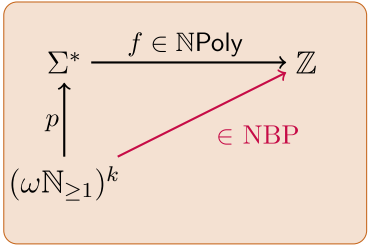
Natural Binomial Polynomials: positive linear combinations of products of binomials

Theorem: $\forall f, \forall p, \exists \omega \in \mathbb{N}, \exists K \in \mathbb{N}$ such that



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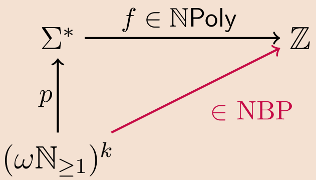
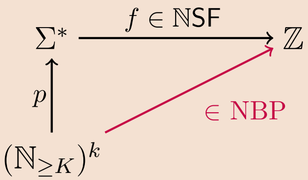
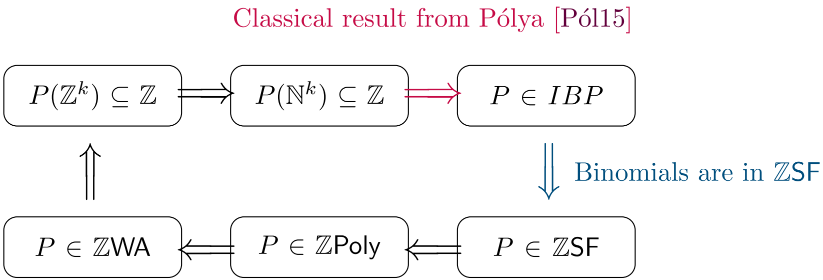




Back to $\mathbb{Z}$

Let $P \in \mathbb{Q}[\vec{X}]$

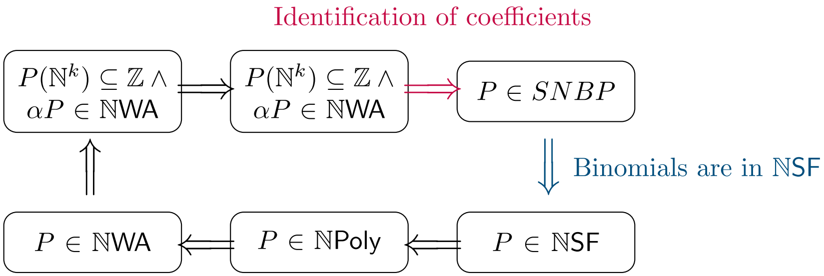


Integer Binomial Polynomials: integer linear combinations of products of binomials.

$$\binom{|w|}{\ell} = \# \left(\bigwedge_{1 \leq i \neq j \leq \ell} x_i \neq x_j \right) (w)$$

And now for $\mathbb{N}$

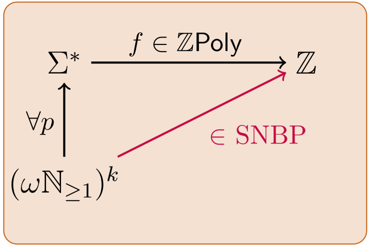
Let $P \in \mathbb{Q}[\vec{X}]$

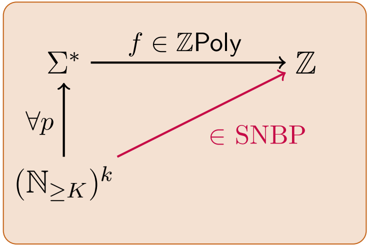


Strongly Natural Binomial Polynomials: Natural Binomial Polynomials, stable under partial evaluations

Tying the knot: commutative input

Commutative Input





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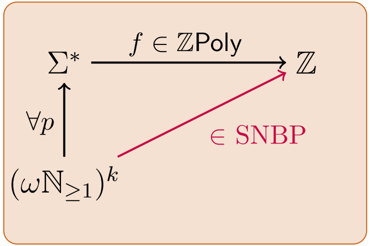
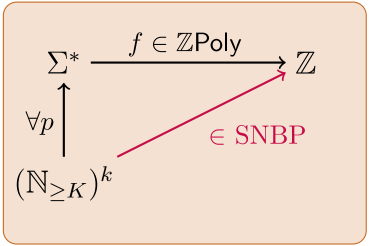
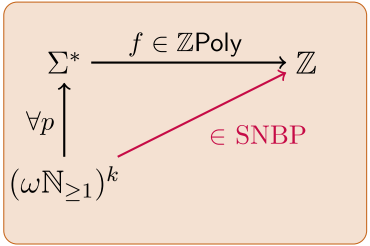


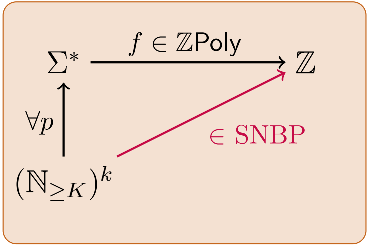


$\exists \omega \Updownarrow$ $\Updownarrow \exists K$

$f \in \mathbb{N}\mathsf{Poly}$ $f \in \mathbb{N}\mathsf{SF}$

Commutative Input





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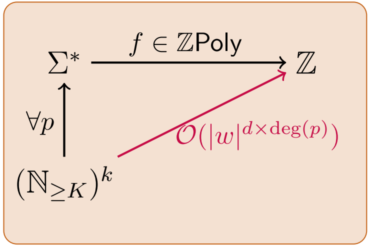
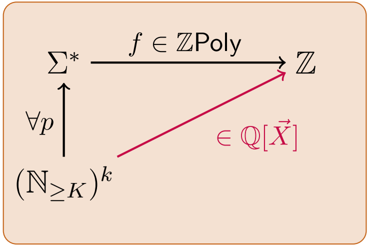
$f \in \mathbb{N}\mathsf{Poly}$ $f \in \mathbb{N}\mathsf{SF}$

Conjecture: true without commutative assumption

Previously, in $\mathbb{Z}$-polyregular functions

Theorems from [CDL23]





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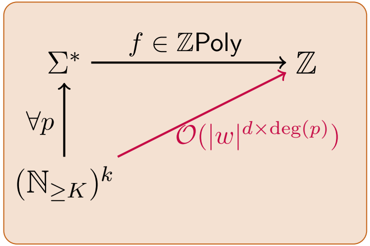

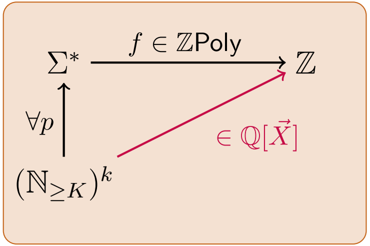

$\exists K \Updownarrow$ $\Updownarrow \exists K$

$$|f(w)| = \mathcal{O}(|w|^d) \qquad f \in \mathbb{Z}\mathsf{SF}$$

Previously, in $\mathbb{Z}$-polyregular functions

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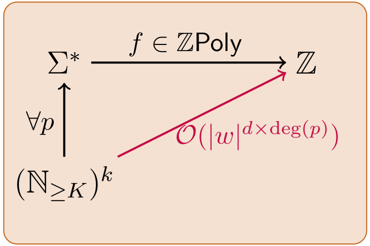

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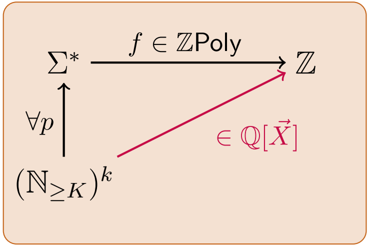
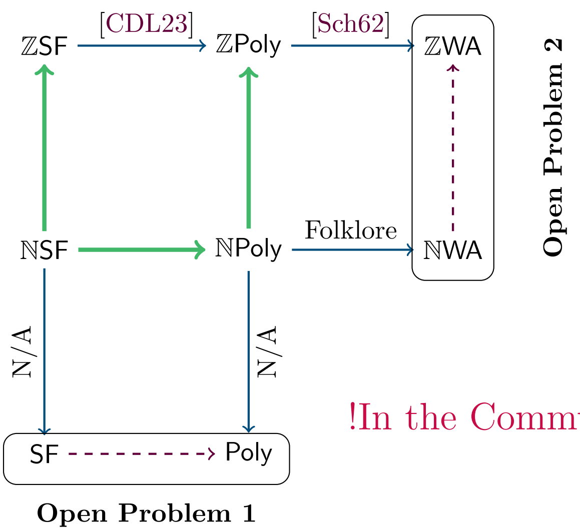


$\Updownarrow \exists K$

$$f \in \mathbb{Z}\mathsf{SF}$$

Consequence: $\mathbb{N}\mathsf{SF} = \mathbb{N}\mathsf{Poly} \cap \mathbb{Z}\mathsf{SF}$ in the commutative setting!

Conclusion



Towards Series: define *aperiodic* ℤWA and ℕWA (but not like [DG05; DG19]).

Spectrum in $\mathbb{R}_{\geq 0}$, aperiodic finite quotients, pumping arguments

Dropping Commutation: attempted a *WQO* approach building a canonical model [Lop24].

Using aperiodic approximations?

!In the Commutative Setting!

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