Labelled Well Quasi Ordered Classes of Bounded Linear Clique-Width

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LaBRI
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Well Quasi Orderings

Every infinite sequence of elements contains an increasing pair.

What is a well-quasi-order (WQO)?

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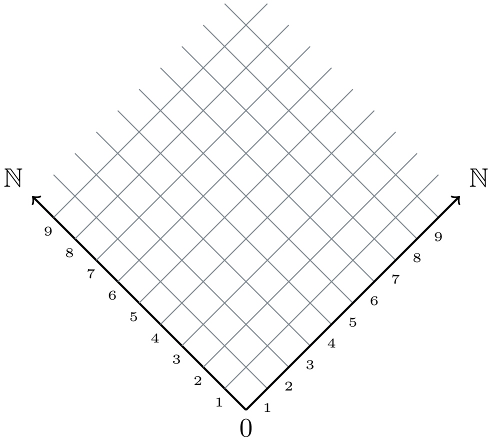

Well-Quasi-Order over $\mathbb{N} \times \mathbb{N}$

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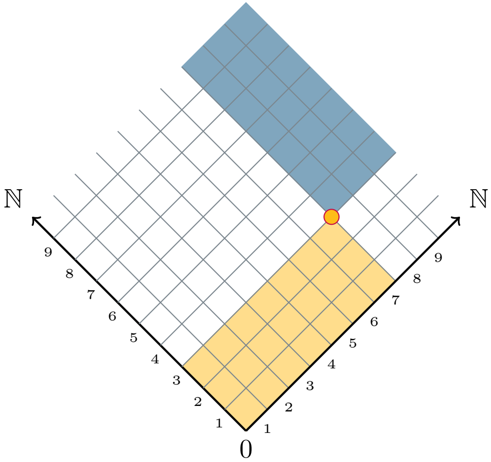

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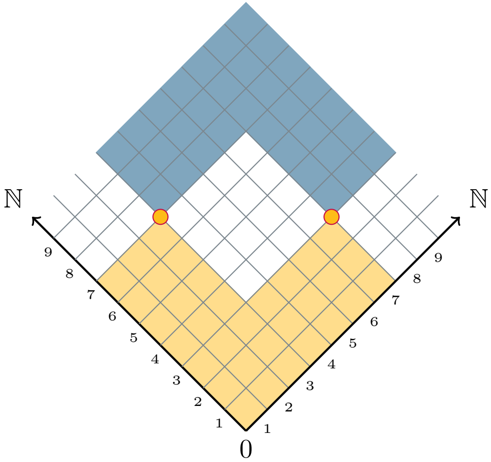

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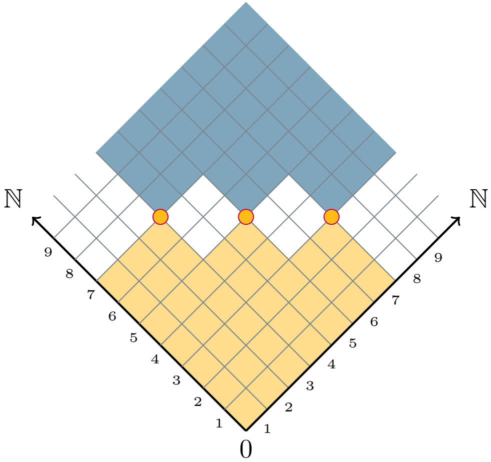

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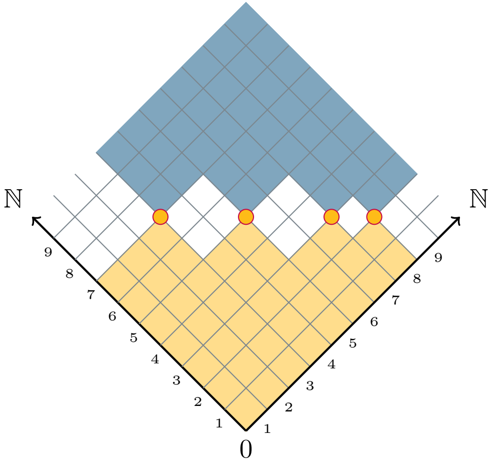

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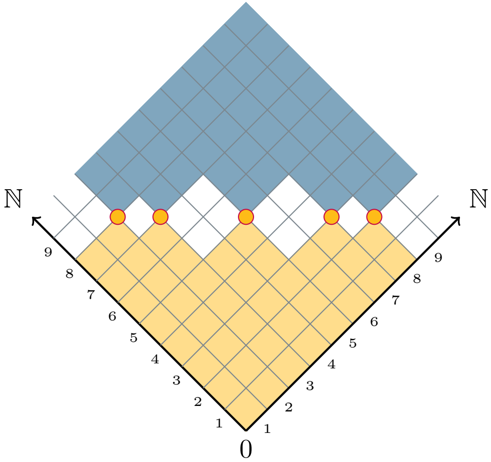

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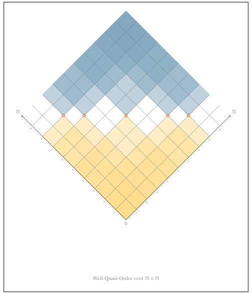

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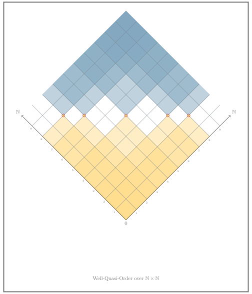

Theory **and** *Practice*

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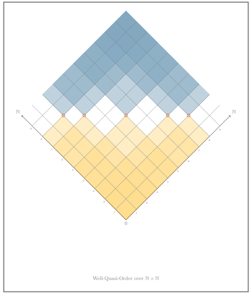

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Verification of Well-Structured-Transition-Systems [Abd+96]

Karp-Miller Coverability [KM69]

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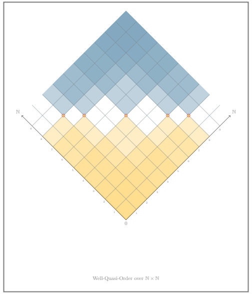

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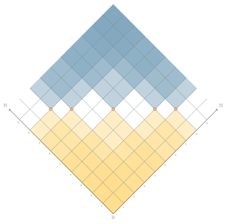

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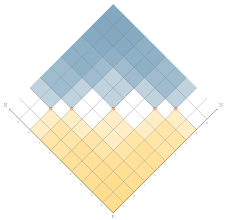

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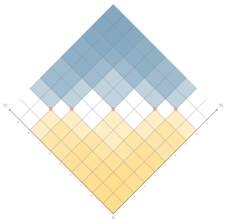

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$$(X, \leq_X) \times (Y, \leq_Y)$$

$$(X, \leq_X) + (Y, \leq_Y)$$

$$(X, \leq_X) \subseteq (Y, \leq_Y)$$

Algebraic Operations

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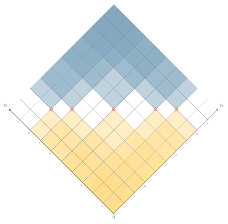

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$$(\mathcal{P}_{\text{fin}}(X), \leq_{\text{hoare}})$$

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Set Functors

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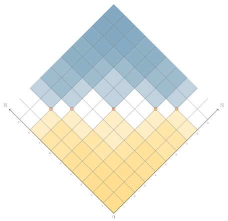

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Algebraic Operations	Set Functors	Structural Functors
$(X, \leq_X) \times (Y, \leq_Y)$	$(\mathcal{P}_{\text{fin}}(X), \leq_{\text{hoare}})$	$(X^*, \leq_{\text{subword}})$ [Hig52]
$(X, \leq_X) + (Y, \leq_Y)$	$(\mathcal{M}^\diamond(X), \leq_{\text{multiset}})$	$(\mathsf{Trees}(X), \leq_{\text{subtree}})$ [Kru72]
$(X, \leq_X) \subseteq (Y, \leq_Y)$		$(\mathsf{Graphs}(X), \leq_{\text{minor}})$ [RS04]

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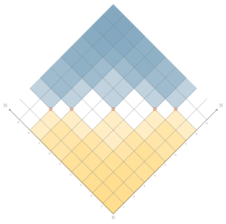

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Structural Functors

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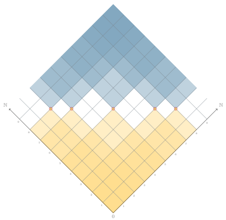

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Structural Functors

Some of these are *inductive* datatypes [Lop23; Fre20], but we want *more*!
Also, can we avoid *minimal bad sequence arguments* [Nas65]?

Classes of Structures...

... as Operators

Structures as Operators

Classes of Structures...

... as Operators

Higman's Subword Embedding

b e a u

Classes of Structures...

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Higman's Subword Embedding

b o r d e a u x

b e a u

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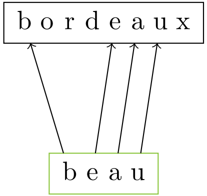
Higman's Subword Embedding



$$\sigma = \{(\leq, 2)\}$$

Classes of Structures...

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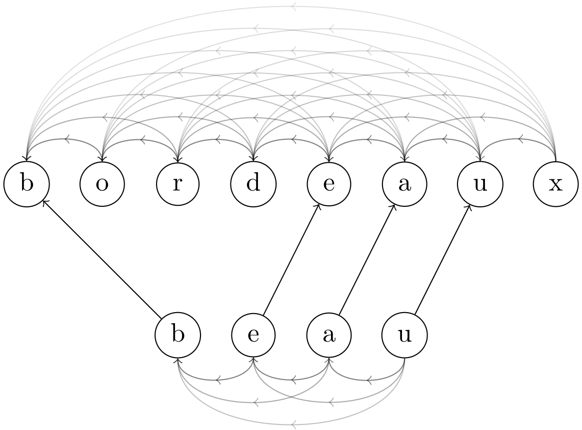

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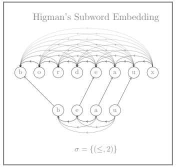

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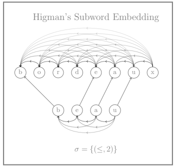
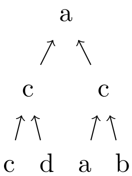


Kruskal's Tree Embedding



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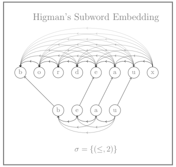

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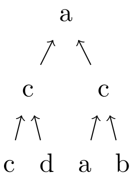

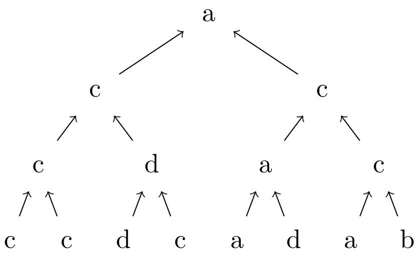

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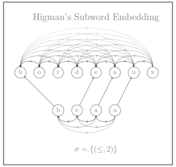

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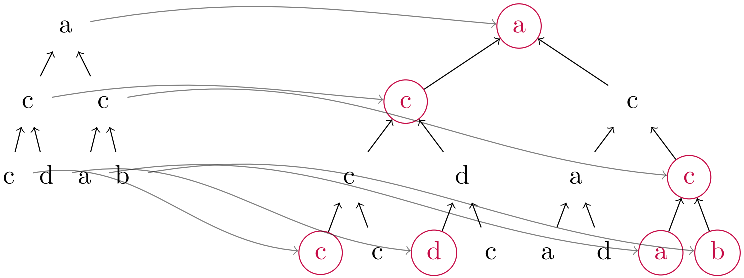

$$\sigma = \{(\leq, 2), (\text{leastCommonAncestor}, 3)\}$$

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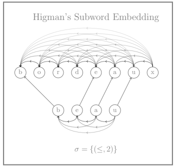

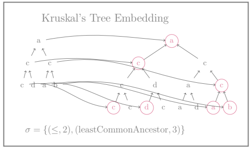

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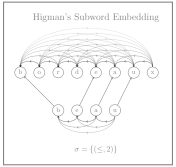

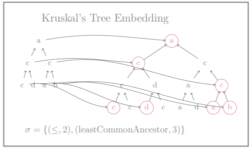

In General

$\mathcal{C}$ a class of finite structures

$\mathsf{Label}_X(\mathcal{C})$ the X-labelled structures in $\mathcal{C}$

$\mathfrak{A} \leq_X \mathfrak{B}$ if there exists a monotone embedding

$(X, \leq)$ WQO $\implies$ $(\mathsf{Label}_X(\mathcal{C}), \leq_X)$ is a WQO?

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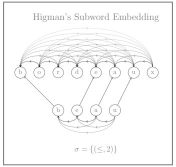

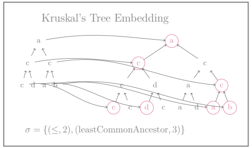

In General	This talk
$\mathcal{C}$ a class of finite structures	$\mathcal{C}$ is a class of **finite undirected graphs**
$\mathsf{Label}_X(\mathcal{C})$ the X-labelled structures in $\mathcal{C}$	$\mathsf{Label}_X(\mathcal{C})$ is a class of **X-labelled-graphs**
$\mathfrak{A} \leq_X \mathfrak{B}$ if there exists a monotone embedding	$\leq_X$ is the **induced subgraph** relation
$(X, \leq)$ WQO $\implies$ $(\mathsf{Label}_X(\mathcal{C}), \leq_X)$ is a WQO?	Is the class of graphs a WQO-operator?

Pouzet Conjectures

Graph Theory

Is
$\mathcal{C}$
WQO?

Operators

Is
$\mathcal{C}$
+ WQO
set of labels
WQO?

Pouzet Conjectures

Graph Theory	?	Model Theory	Operators
Is $\mathcal{C}$ WQO?	Is $\mathcal{C}$ + 2 labels WQO?	Is $\mathcal{C}$ + finite set of labels WQO?	Is $\mathcal{C}$ + WQO set of labels WQO?

Pouzet Conjectures

Graph Theory	?	Model Theory	Operators
Is $\mathcal{C}$	⇐ Is $\mathcal{C}$ + 2 labels	⇐ Is $\mathcal{C}$ + finite set of labels	⇐ Is $\mathcal{C}$ + WQO set of labels
WQO?	WQO?	WQO?	WQO?

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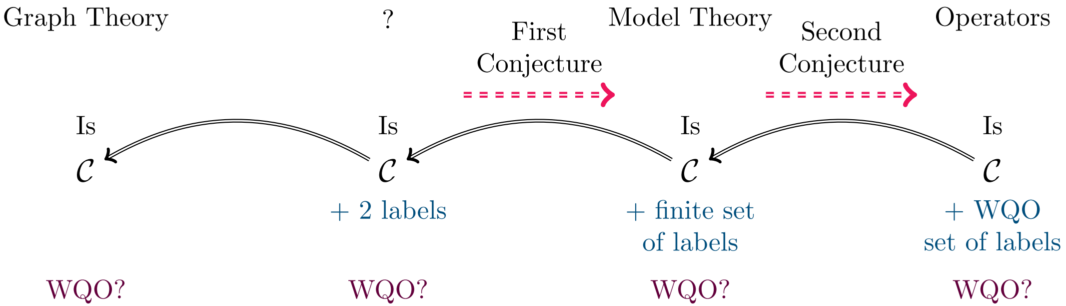
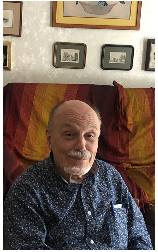


Pouzet Conjectures

Graph Theory	?	Model Theory	Operators
Is $\mathcal{C}$	Is $\mathcal{C}$ + 2 labels	Is $\mathcal{C}$ + finite set of labels	Is $\mathcal{C}$ + WQO set of labels
WQO?	WQO?	WQO?	WQO?

First Conjecture (between "+ 2 labels" and "+ finite set of labels")

Second Conjecture (between "+ finite set of labels" and "+ WQO set of labels")



First Pouzet Conjecture: $2 = \omega$ [Pou72]

Second Pouzet Conjecture: $\omega = \mathsf{WQO}$

Schmitz's Conjecture: *finite paths* are the only obstruction to 2-WQO

Finite Paths

Finite Paths are WQO but not 2-WQO

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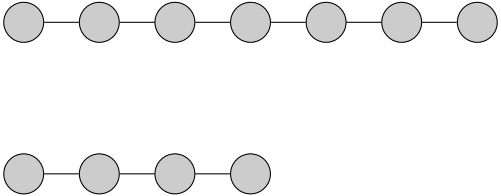

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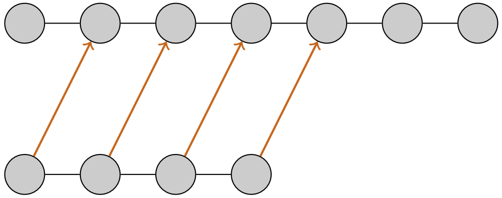

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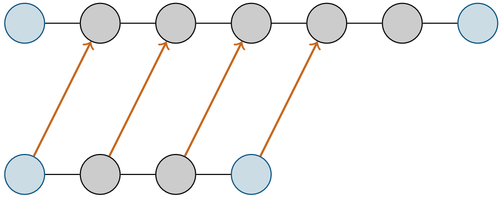

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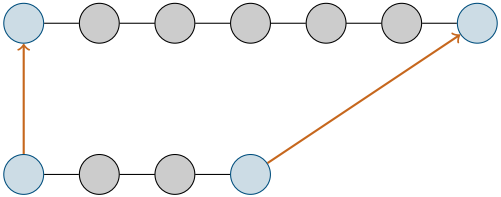
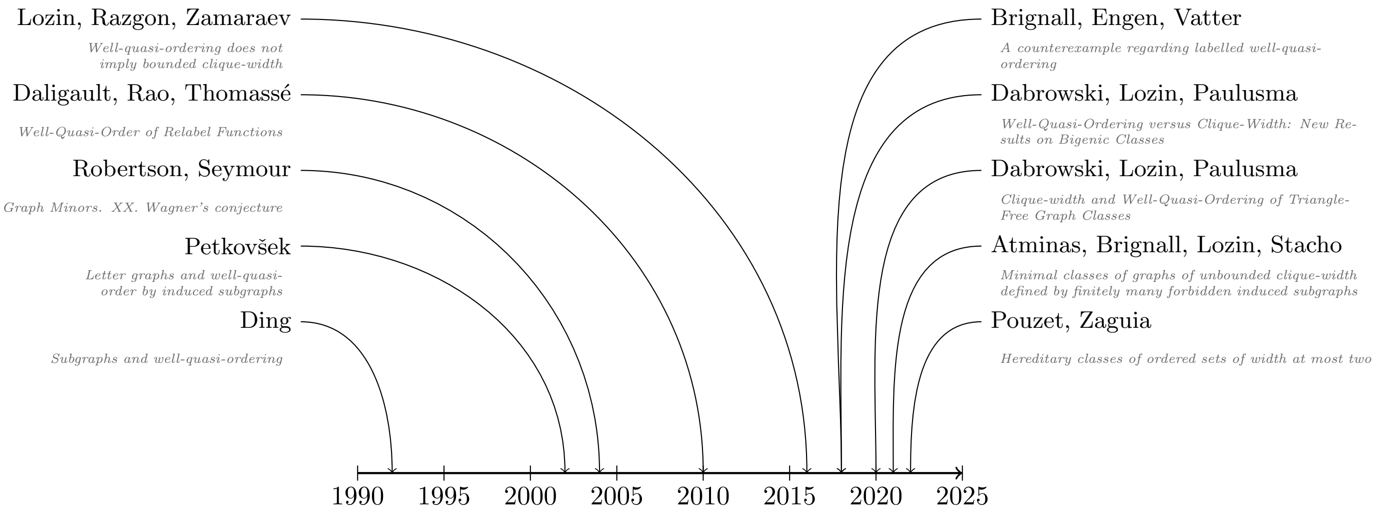


State of the Art

Well Quasi Ordering Graph Classes in 2024



Usual Proof Method

From Graphs To Trees

$$(\mathsf{Trees}, \leq) \xrightarrow{\text{monotone, surjective}} (\mathcal{C}, \subseteq_i)$$

Usual Proof Method

From Graphs To Trees

Nested Higman?

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Kruskal?

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Usual Proof Method

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Typical Example

$$T \in \mathsf{Trees}(\Sigma, \mathbb{P}(\Sigma \times \Sigma)) \longrightarrow G \in \mathcal{C}$$

m-partite cographs

Σ finite with $m = |\Sigma|$

order = Kruskal

$$V(G) = \text{leaves of } T$$

$$(x, y) \in E(G) \iff (c(x), c(y)) \in \mathsf{lca}(x, y)$$

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Bounded Clique Width

A Logical Approach to Graph Decompositions

Bounded Clique Width

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$$\mathsf{Trees} \xrightarrow{\mathsf{MSO}\text{ definable, surjective}} \mathcal{C}$$

Bounded Clique Width

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Introduced by Courcelle [Cou91; Cou94; CER93] with the idea of generalizing **regular languages** to graphs.

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$\varphi_{\text{dom}}(x)$: selects vertices of the graph

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$\varphi_\Delta = \top$, $\varphi_{\text{dom}}(x) = \top$, $\varphi_{\text{edge}}(x, y) = \top$: builds all cliques

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Trees $\xrightarrow{\text{MSO definable, surjective}}$ $\mathcal{C}$

φ_Δ : consider this tree?

$\varphi_{\text{dom}}(x)$: selects vertices of the graph

$\varphi_{\text{edge}}(x, y)$: selects edges of the graph

$\varphi_\Delta = \top$, $\varphi_{\text{dom}}(x) = \top$, $\varphi_{\text{edge}}(x, y) = \top$: builds all cliques

- Gives a finite representation of graph classes
- Is a well-studied parameter
- Is conjectured to be tightly related to being labelled-WQO

Bounded Clique Width

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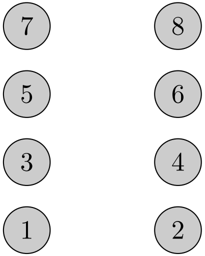
To simplify *everything* consider linear trees, i.e., **finite words**

Bounded **Linear** Clique-Width

b	o	r	d	e	a	u	x
1	2	3	4	5	6	7	8



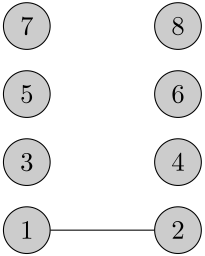
$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

Bounded **Linear** Clique-Width

b o r d e a u x

1 2 3 4 5 6 7 8

x y



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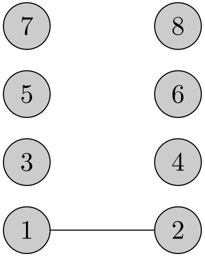
Bounded **Linear** Clique-Width

b o r d e a u x

1 2 3 4 5 6 7 8

x y



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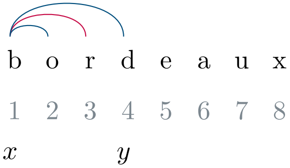

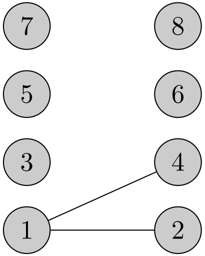

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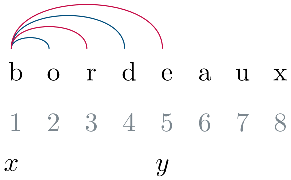
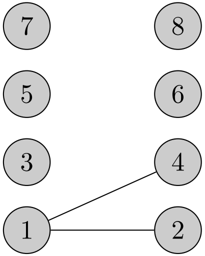
Bounded **Linear** Clique-Width





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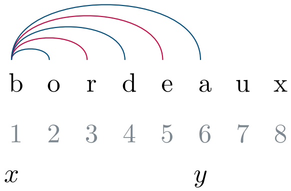
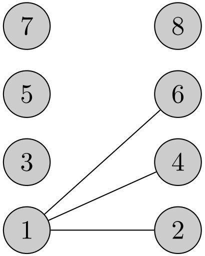
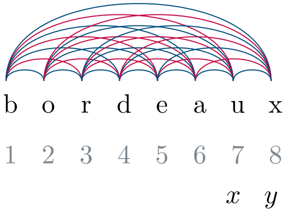


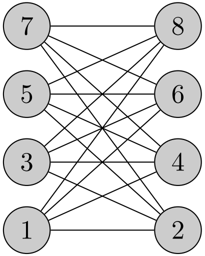


$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(1, 6) \vee \varphi(6, 1) = \top$$

Bounded **Linear** Clique-Width





$$\varphi(x, y) = \text{isOdd}(x) \wedge \text{isEven}(y)$$

$$\varphi(7, 8) \vee \varphi(8, 7) = \top$$

The Automata Toolbox

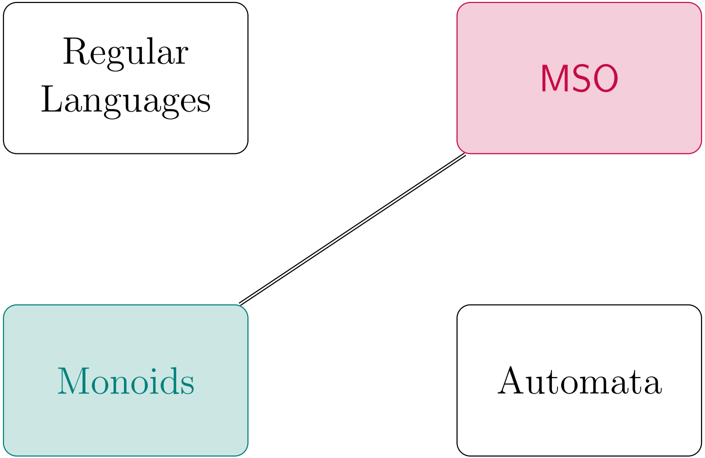
Regular Languages

MSO

Monoids

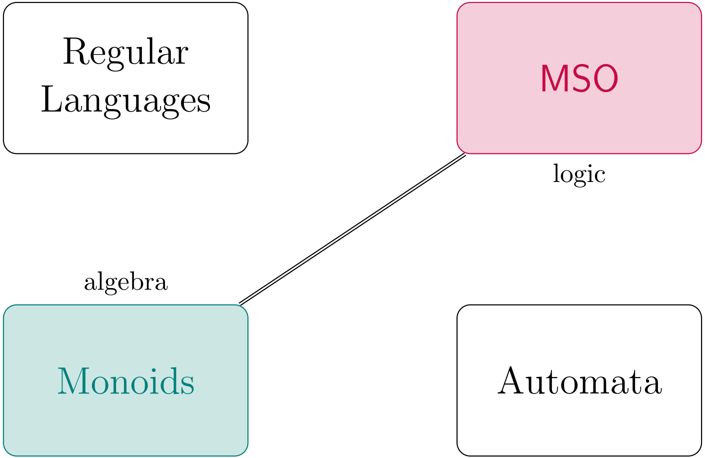
Automata

The Automata Toolbox



The Automata Toolbox

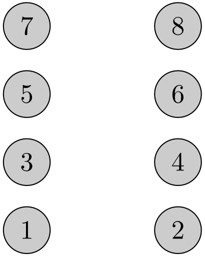
$\mathrm{isOdd}(x) \wedge \mathrm{isEven}(y)$



$M = (\mathbb{Z}/2\mathbb{Z}, +)$

$\mu \colon \Sigma \to M$

Bounded **Linear** Clique-Width



b o r d e a u x

1 2 3 4 5 6 7 8

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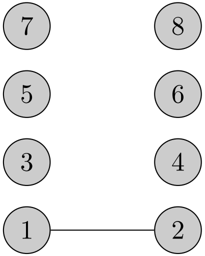

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$\vdash$ —— x - y ———————— $\dashv$

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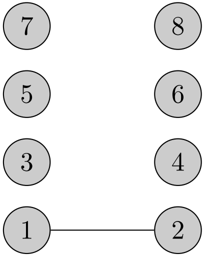
$$(\mu(w[1:1]), \mu(w[1:2]), \mu(w[2:|w|])$$

$$= (0, 0, 0)$$

$$\in P? \ \top$$

Bounded **Linear** Clique-Width



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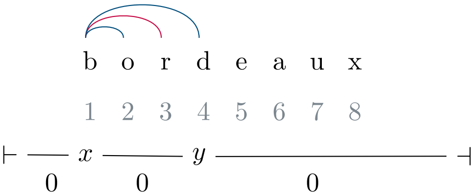

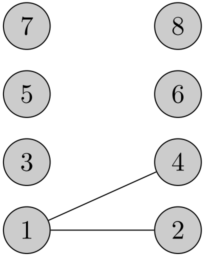

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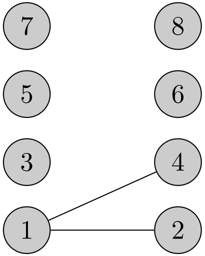

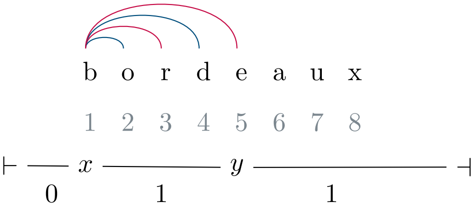

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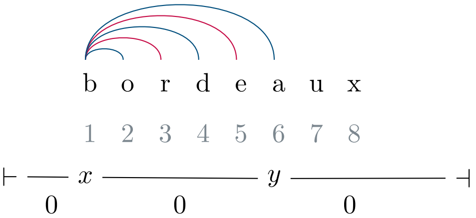

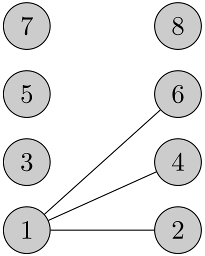

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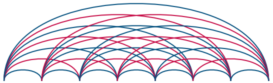

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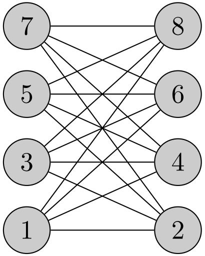

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Simon's Factorisation(s)

A Ramsey-like fundamental theorem of semigroup theory [Sim90].

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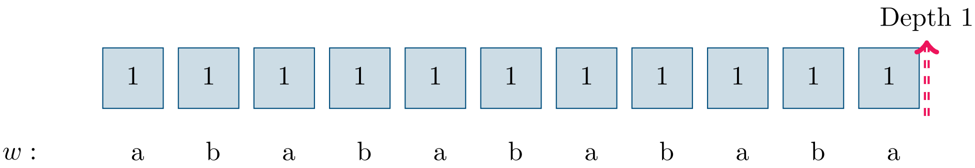

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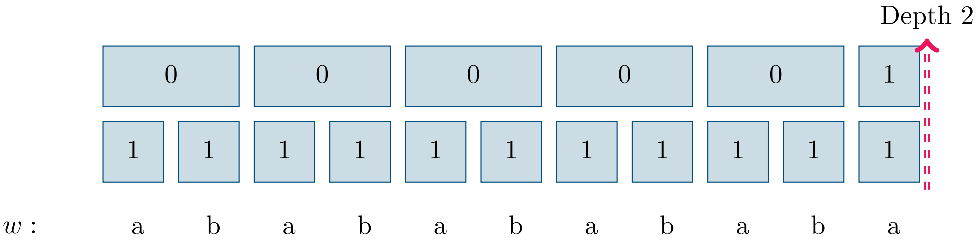

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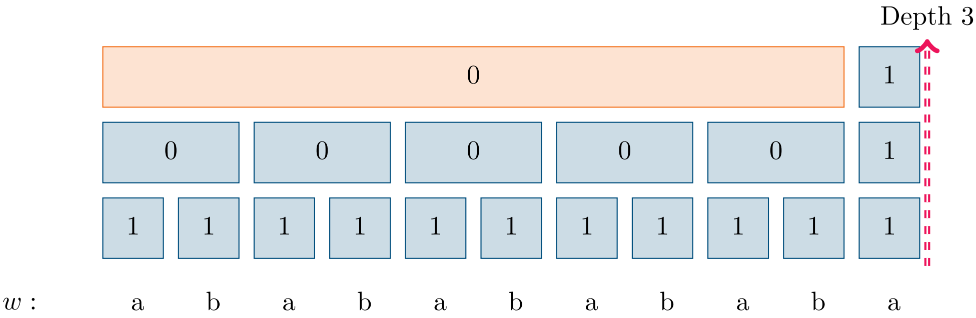

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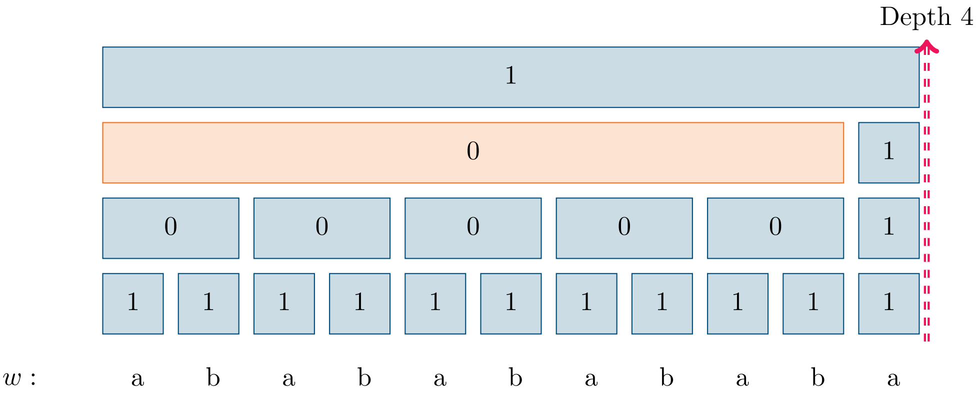

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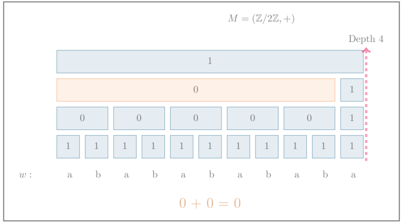

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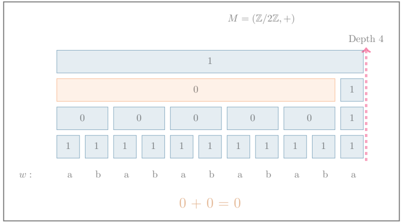

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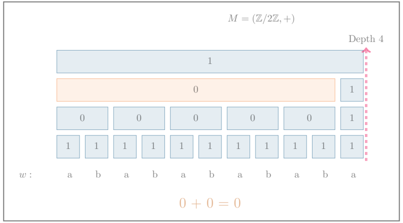

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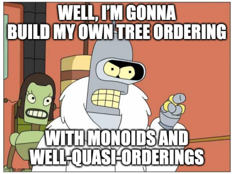

BACK TO THE TREES!

Take Away Food For Thought

If there was only one thing to remember from this talk ...

Good Orders On Factorisations $\simeq$ Well-Quasi-Ordered classes of graphs

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Nested Higman Over Simon's	$\simeq$	Well-Quasi-Ordered classes of graphs

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Gap-Embedding [DT03] $\simeq$ Well-Quasi-Ordered classes of graphs

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Logic + WQO = ♡

The Main Issue

Inserting idempotents does not affect **far away** positions...

... But does affect **close** positions!

$$M = (\{0, 1, +\infty\}, +),\ \mu(\varepsilon) = 0,\ \mu(a) = \mu(b) = 1$$

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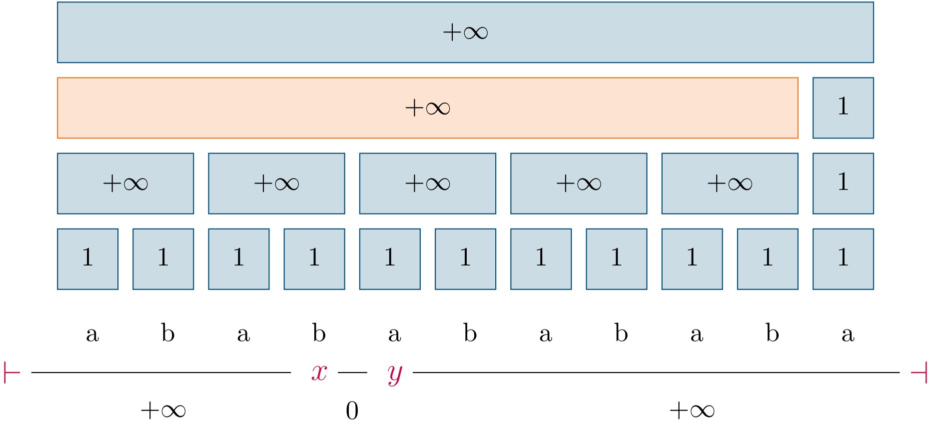

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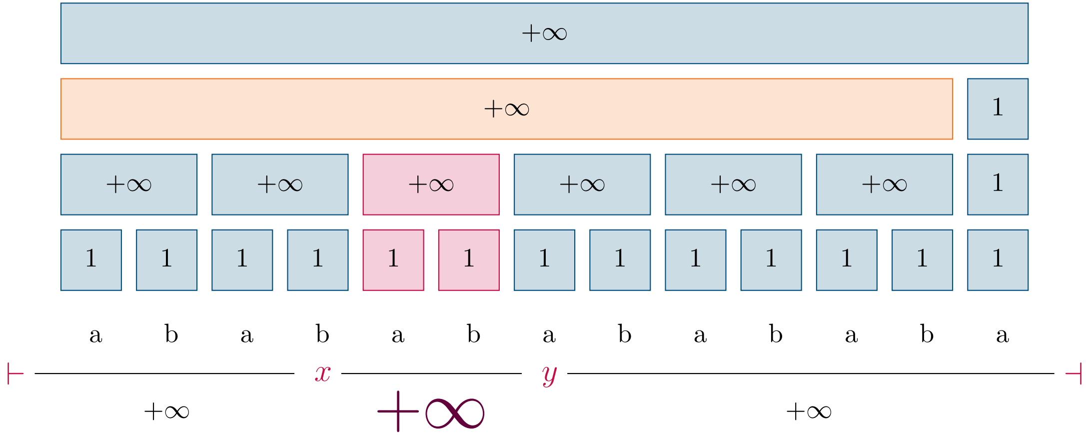

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Conclusion / Ask Me Anything

Theorem 1: Given an MSO transduction, one can **decide** if its *image* is labelled-WQO, and **compute** a k such that k-wqo $\iff$ labelled-wqo.

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In the near future:
Clique-Width (almost written up, using [Lop23]),
Schmitz's Conjecture (almost done for clique-width),
labelled-wqo implies ω-categorical (in progress).

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