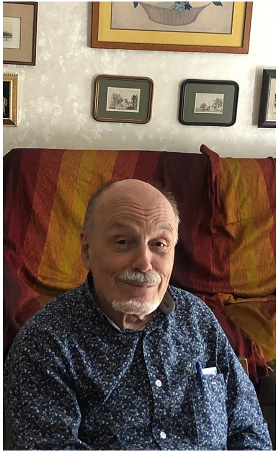
Labelled Well Quasi Ordered Classes of Bounded Linear Clique-Width

Aliaume Lopez
University of Warsaw

Highlights 2024

https://www.irif.fr/~alopez/



In Memory of

Maurice Pouzet

1945 – 2023

Labelled
Well Quasi Ordered Classes

1

of
Bounded Linear Clique-Width

Well Quasi Orderings

Every infinite sequence of elements contains an increasing pair.

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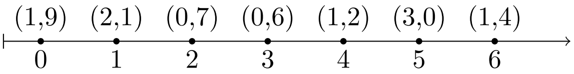

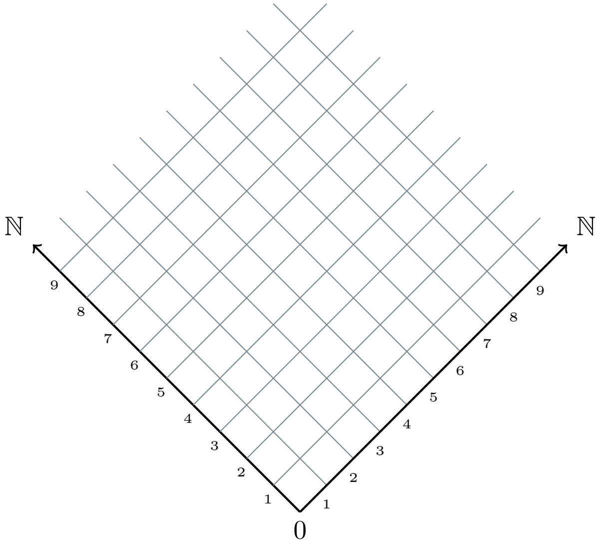

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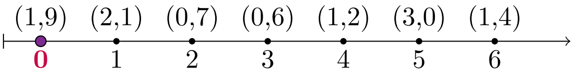

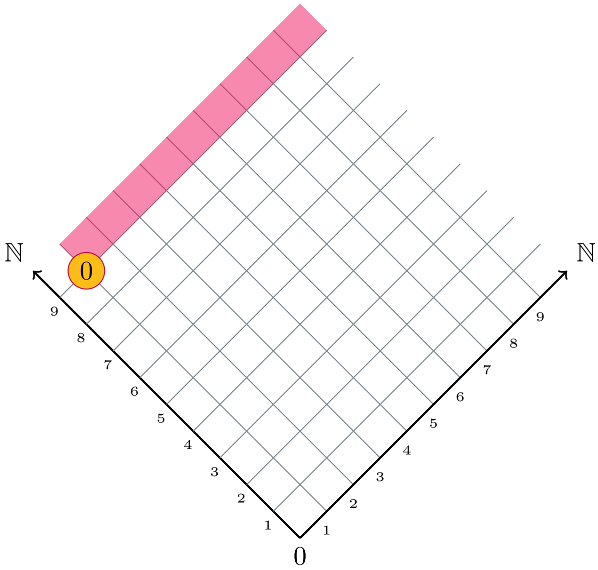

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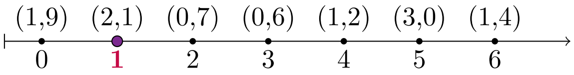

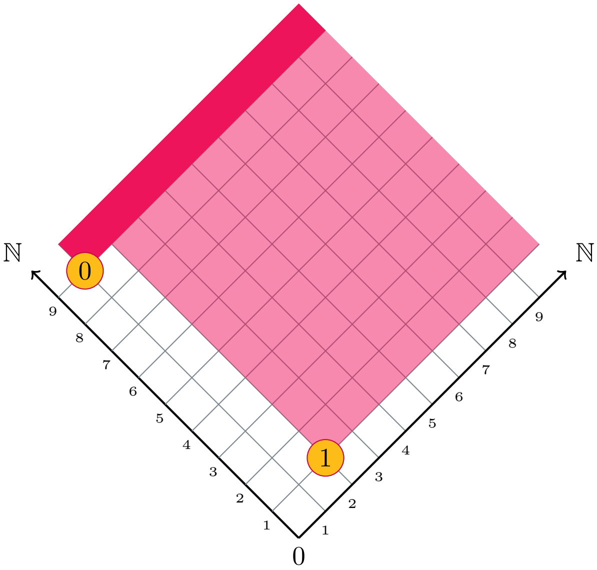

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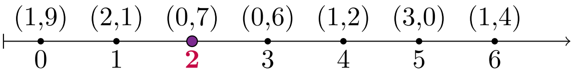

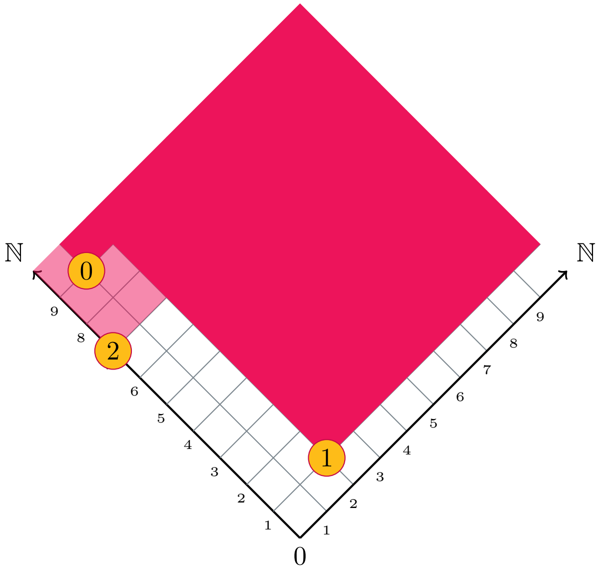

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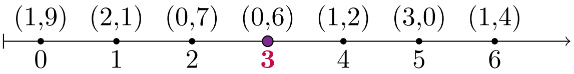

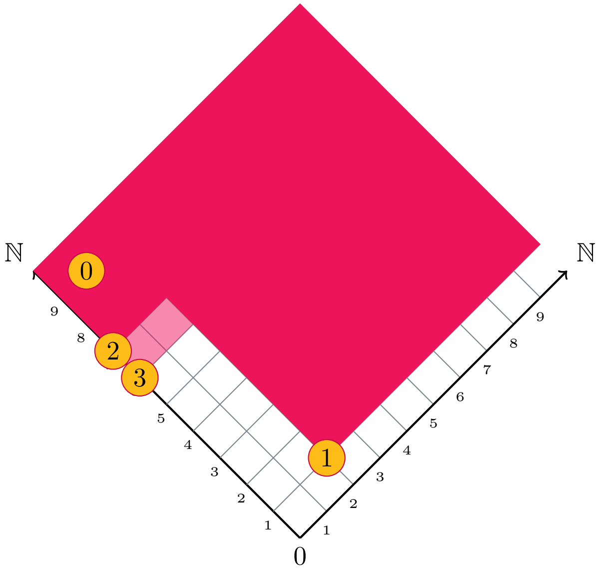

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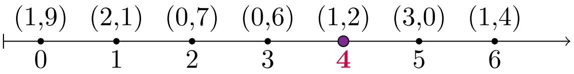

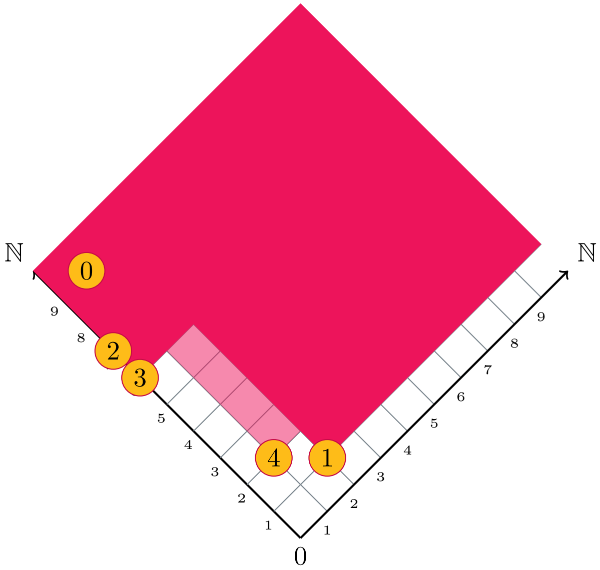

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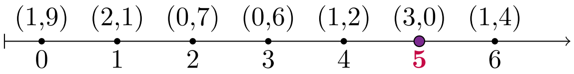

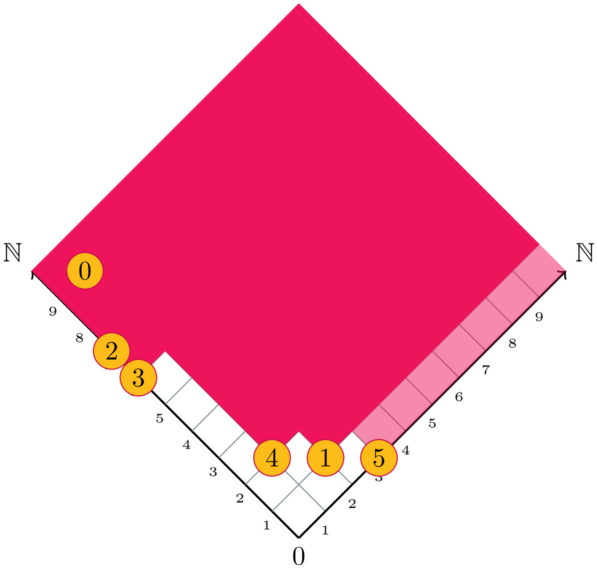

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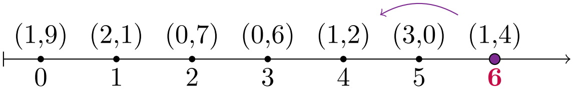

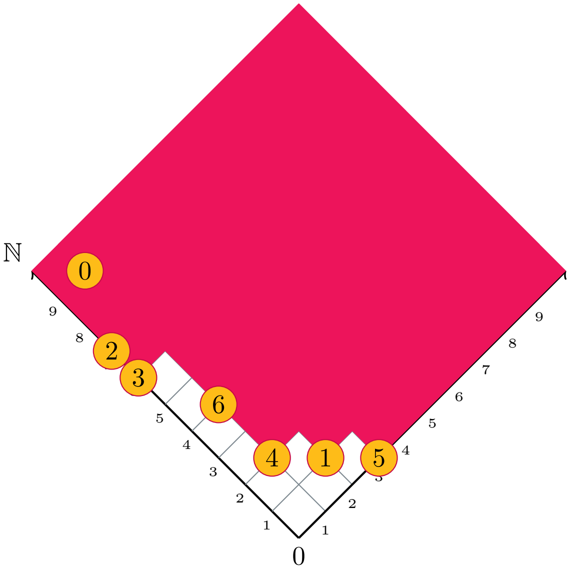

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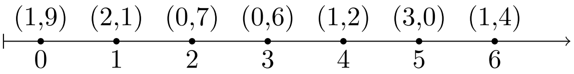
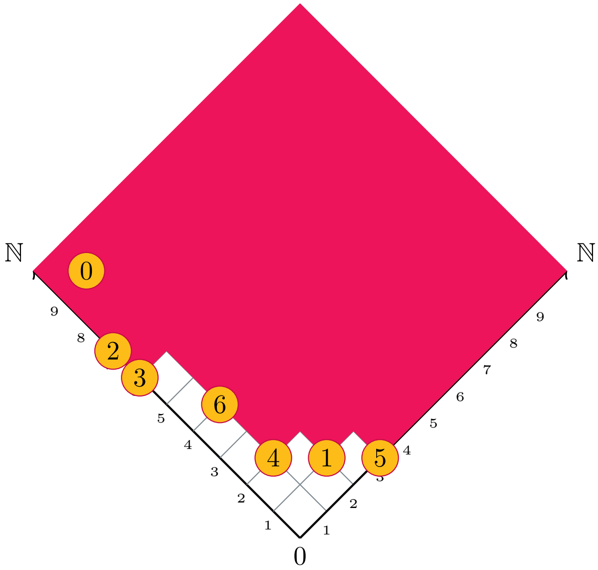




Examples: $\mathbb{N}$, $\mathbb{N}^2$, Σ^*,
Graphs with the Minor Relation [1] . . .

Labelled (2)

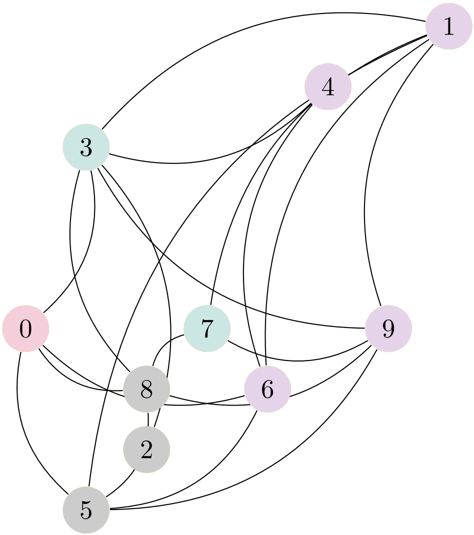
Well Quasi Ordered Classes (1)

of

Bounded Linear Clique-Width

Labelled

Finite undirected graphs with labels in an *ordered* set L, equipped with the induced subgraph relation.



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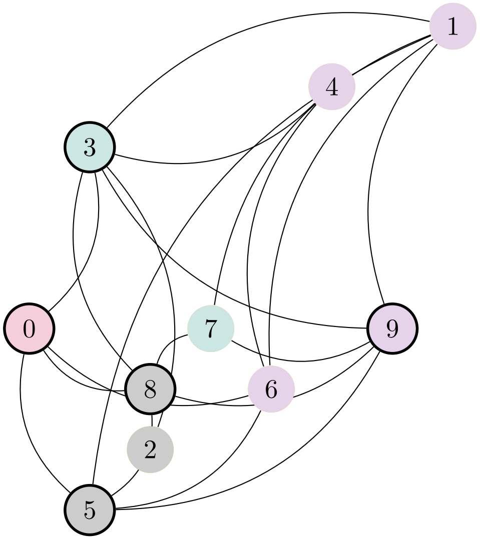

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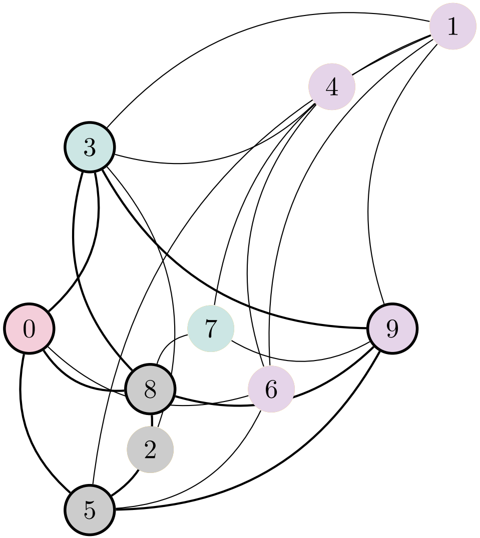

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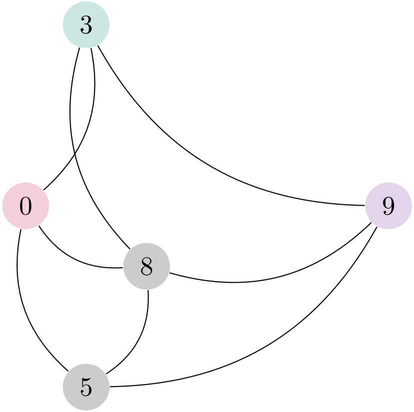

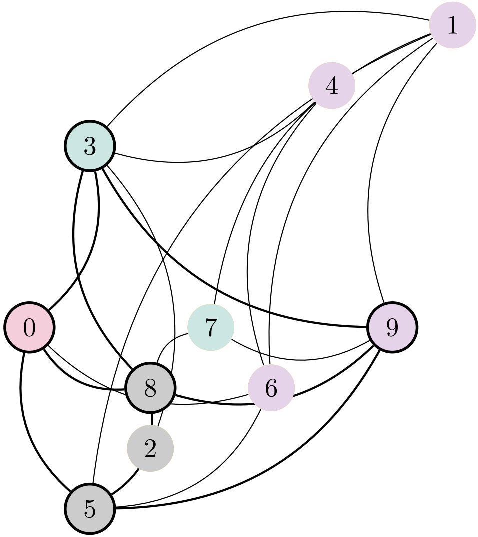

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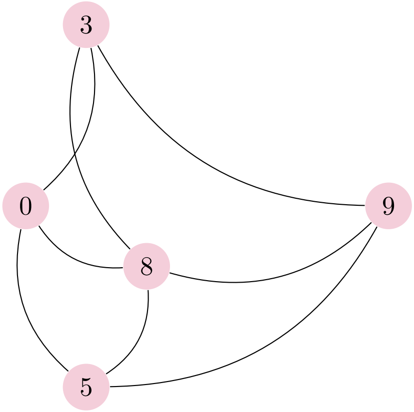
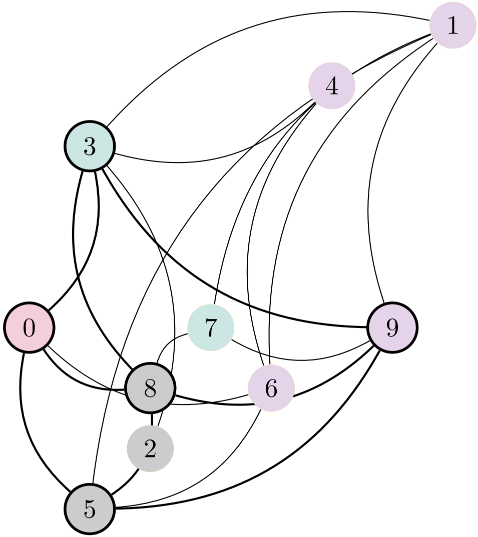


$\subseteq_i$



Labelled Well-Quasi-Ordered Classes

For a class $\mathcal{C}$ of finite undirected graphs, we can ask the following questions.

Is

$$\mathcal{C}$$

WQO?

Labelled Well-Quasi-Ordered Classes

For a class $\mathcal{C}$ of finite undirected graphs, we can ask the following questions.

Is $\mathcal{C}$	Is $\mathcal{C}$	Is $\mathcal{C}$	Is $\mathcal{C}$
	+ 2 labels	+ finite set of labels	+ WQO set of labels
WQO?	WQO?	WQO?	WQO?

Labelled Well-Quasi-Ordered Classes

For a class $\mathcal{C}$ of finite undirected graphs, we can ask the following questions.

Is $\mathcal{C}$	Is $\mathcal{C}$ + 2 labels	Is $\mathcal{C}$ + finite set of labels	Is $\mathcal{C}$ + WQO set of labels
WQO?	WQO?	WQO?	WQO?

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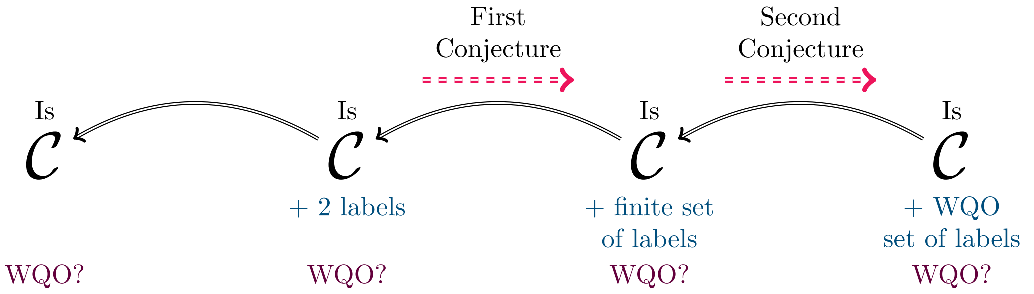


Labelled 2

Well Quasi Ordered Classes 1

of

Bounded Linear Clique-Width 3

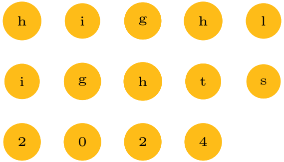
Bounded Linear Clique-Width

$$\varphi : \Sigma^* \xrightarrow{\mathsf{MSO}} \text{Graphs}$$

Bounded Linear Clique-Width

$$\varphi : \Sigma^* \xrightarrow{\text{MSO}} \text{Graphs}$$

h i g h l i g h t s 2 0 2 4

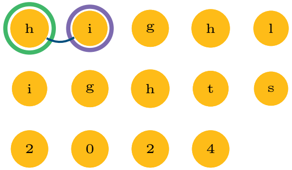


Bounded Linear Clique-Width

$$\varphi : \Sigma^* \xrightarrow{\mathsf{MSO}} \text{Graphs}$$

h i g h l i g h t s 2 0 2 4

$$\underset{x}{\uparrow}\ \underset{y}{\uparrow}$$



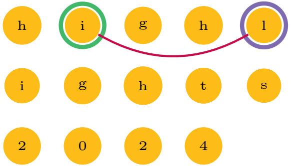
$\varphi(x, y) = \mathsf{True}$

Bounded Linear Clique-Width

$$\varphi : \Sigma^* \xrightarrow{\text{MSO}} \text{Graphs}$$

h i g h l i g h t s 2 0 2 4

x y



$\varphi(x, y) =$ False

Bounded Linear Clique-Width

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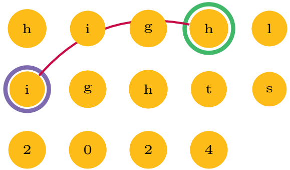
h i g h l i g h t s 2 0 2 4

$\uparrow \quad \uparrow$

$x \quad y$



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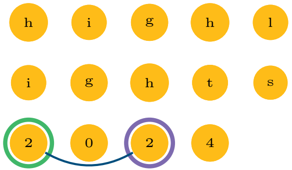

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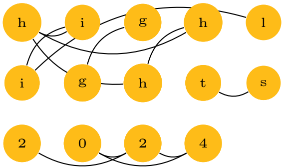

Labelled ② Well Quasi Ordered Classes ① of Bounded Linear Clique-Width ③

Results

$\forall \varphi \in \mathsf{MSO}, \exists$(computable)$k \in \mathbb{N}$ such that

Is $\varphi(\Sigma^*)$ WQO? $\Longleftarrow$ Is $\varphi(\Sigma^*)$ + k labels WQO? $\xLeftrightarrow{\text{First Theorem}}$ Is $\varphi(\Sigma^*)$ + finite set of labels WQO? $\xLeftrightarrow{\text{Second Theorem}}$ Is $\varphi(\Sigma^*)$ + WQO set of labels WQO?

https://arxiv.org/abs/2405.10894

Labelled 2

Well Quasi Ordered Classes 1

of

Bounded Linear Clique-Width 3

References

[1] Neil Robertson and Paul D. Seymour. Graph minors. xx. wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004. Special Issue Dedicated to Professor W.T. Tutte.