TD 3: CTL, CTL* Solutions

Exercice 1

The two formulæ are equivalent in all cases but 4.

Exercice 2

$[\![\mathsf{AG}\, q]\!]$ contains only 5, $\mathsf{G}\, p = 4(34)^\omega + 3(43)^\omega$ hence $[\![\mathsf{EF}\,\mathsf{G}\, p]\!]$ contains all nodes but 5 (3 or 4 are reachable from any other node). Thus $[\![(\mathsf{EF}\,\mathsf{G}\, p)\ \mathsf{U}\ (\mathsf{AG}\, q)]\!]$ is the set of all paths that end in 5. This means $[\![\mathsf{FA}((\mathsf{EF}\,\mathsf{G}\, p)\ \mathsf{U}\ (\mathsf{AG}\, q))]\!]$ too is the set of all paths that end in 5. On the other hand $[\![\mathsf{X}\, q]\!]$ is the set of paths which second node is either 4 or 5 (it cannot be 1 because no node has a transition to 1). Hence $[\![\varphi]\!]$ is the set of nodes from which all paths either end in 5 or have 4 as a second node, meaning $[\![\varphi]\!] = \{2, 3, 5\}$.

Exercice 3

1. Suppose $s \in [\![\mathsf{EG}\,\varphi]\!]$. Thus there exists a path $sv_0v_1...$ in M such that for $s \models \varphi$ and for all n $v_n \models \varphi$. Since S is finite there exists $i < j$ such that $v_i = v_j$. Hence v_i belongs to a non-trivial strongly connected component in M_φ, and there exists a path $s \to^* v_i$ in M_φ. Suppose now there exists a strongly connected component C in M_φ and a node $t \in C$ such that there exists a path $s \to^* t$ in M_φ. Then there exists an infinite path in M $s \to^* t(\to^* t)^\omega$ such that all nodes along the path satisfy φ, and thus $s \models \mathsf{EG}\,\varphi$.

2. To compute $[\![\mathsf{EG}\,\varphi]\!]$, we compute M_φ and its strongly connected components, and then the nodes from which one of these SCCs is reachable in M_φ. Each step takes linear time (for the computation of SCCs, use Tarjan's algorithm), hence the complexity is $\mathcal{O}(n)$.

Exercice 4

1. $\mathsf{E}((a_1 \wedge a_2)\ \mathsf{U}\ (b_1 \wedge \mathsf{E}(a_2\ \mathsf{U}\ b_2))) \vee \mathsf{E}((a_1 \wedge a_2)\ \mathsf{U}\ (b_2 \wedge \mathsf{E}(a_1\ \mathsf{U}\ b_1)))$

2. $$\bigvee_{\sigma \in S_n} \mathsf{E}(\varphi \wedge \bigwedge_{i=1\ldots,n} \psi_{\sigma(i)})\ \mathsf{U}\ (\psi'_{\sigma(1)} \wedge (\varphi \wedge \bigwedge_{i=2,\ldots,n} \mathsf{E}(\psi_{\sigma(i)}\ \mathsf{U}\ (\psi'_{\sigma_2} \wedge \cdots \wedge \mathsf{EG}\,\varphi)\ldots)))$$

   The new formula has exponential size compared to the old one.

3. $[b \wedge \mathsf{E}(\mathsf{X}(a \wedge (b\ \mathsf{U}\ c)))] \vee [c \wedge \mathsf{E}\,\mathsf{X}\, a]$

4. It suffices to apply the translation of question 2 to

$$\bigvee_{I\subseteq[\![1,n]\!]} \bigwedge_{i\in I} \psi_i' \ \wedge\ \bigwedge_{i\notin I} \psi_i \ \wedge\ \varphi' \ \wedge\ \mathsf{E}(\mathsf{X}(\varphi \ \wedge\ \mathsf{E}(\bigwedge_{i\notin I}(\psi_i \,\mathsf{U}\, \psi_i') \wedge \mathsf{G}\,\varphi')))$$

Again, there is an exponential blow up in the size of the formula.

5. We find $\neg\, \mathsf{E}(\mathsf{X}\, b \ \wedge\ \mathsf{G}(\neg a \wedge d)) \wedge \neg\, \mathsf{E}(\mathsf{G}\, c) \wedge \neg\, \mathsf{E}(c \,\mathsf{U}\, \neg d)$.