The Escardò-Lawson-Simpson Construction

Jean Goubault-Larrecq

ENS Cachan

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Context

- **Top** is not Cartesian-closed
- . . . but some full subcategories are,

  e.g., **k-spaces**

  [Brown 61, 63; Steenrod 67; Kelley]
- . . . an instance of a more general construction

  [Escardò-Lawson-Simpson 04]

- We give a **categorical generalization**
- . . . which we apply to **streams** [Krishnan 08].

Outline

The $\mathbf{Map}_{\mathcal{C}}$ Category [Escardò-Lawson-Simpson 04]

Fix a class $\mathcal{C}$ of spaces (e.g., compact Hausdorff spaces).

- $\mathcal{C}$-*probe* (on X) =

  continuous map $C \xrightarrow{k} X$, for some $C \in \mathcal{C}$
- $f : X \to Y$ is $\mathcal{C}$-*continuous*

  iff $f \circ k$ continuous for every $\mathcal{C}$-probe k

Definition ($\mathbf{Map}_{\mathcal{C}}$)

Objects=topological spaces. Morphisms=$\mathcal{C}$-*continuous* maps.

Note: Just like **Top**, with relaxed notion of continuity

(continuous $\Rightarrow$ $\mathcal{C}$-continuous)

The $\mathbf{Map}_{\mathcal{C}}$ Category

Definition (Strongly productive)

$\mathcal{C}$ is *strongly productive* iff:

- every space in $\mathcal{C}$ is exponentiable
- $\mathcal{C}$ is closed under binary products

Note: the exponentiable spaces are the core-compact spaces

(slight generalization of locally compact)

Theorem (Escardò, Lawson, Simpson 2004)

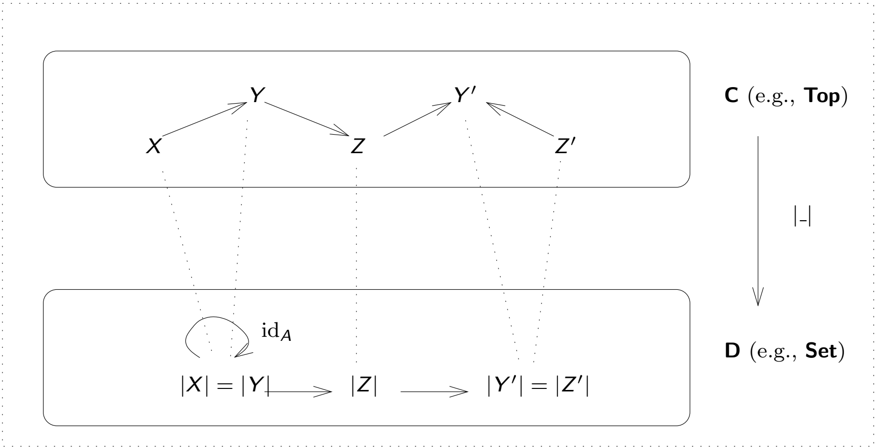
If $\mathcal{C}$ is strongly productive, then $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed.

Many CCCs of topological spaces

We shall see that $\mathbf{Map}_{\mathcal{C}} \cong \mathbf{Top}_{\mathcal{C}}$ CCC of topological spaces:

- $\mathcal{C} =$ all core-compact spaces
  $\Rightarrow$ *largest* such CCC (quotients of core-compact spaces)
- $\mathcal{C} =$ compact Hausdorff spaces
  $\Rightarrow$ quotients of loc. compact spaces
- same as above $+$ (weak) Hausdorff
  k-spaces
- $\mathcal{C} =$ one-point compactification of $\mathbb{N}$
  $\Rightarrow$ sequential spaces
- etc.

Topological functors



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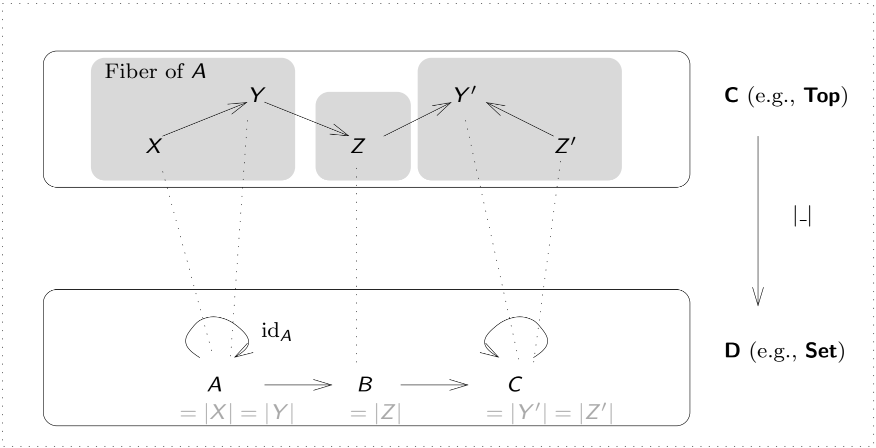

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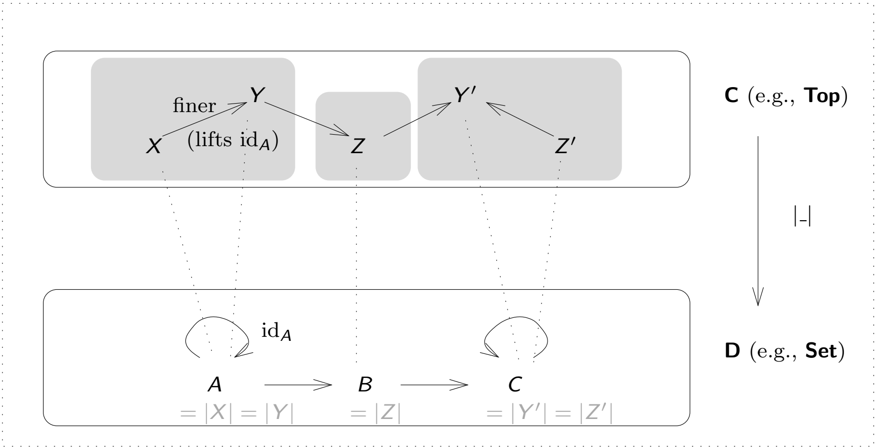


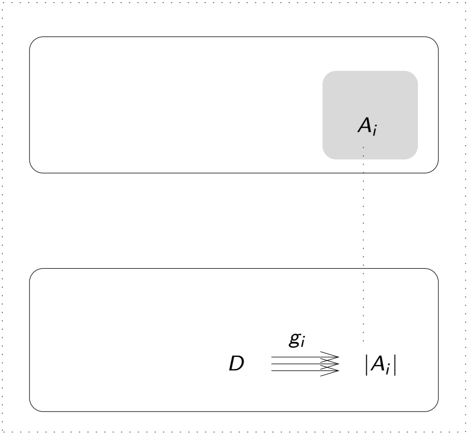
Topological functors

Definition

$|\_| : \mathbf{C} \to \mathbf{D}$ is a *topological* functor iff:

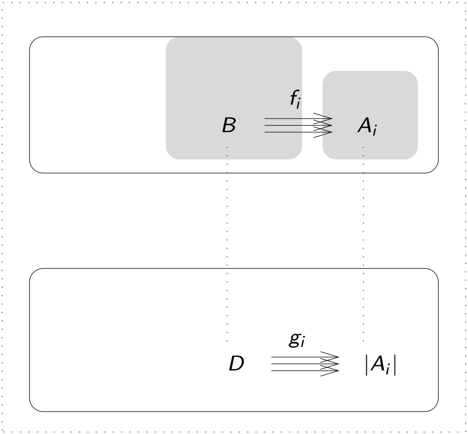
- faithful
- amnestic — “finer” is antisymmetric
- and every $|\_|$-source has a $|\_|$-initial lift — (see next slide)

Every $|\_|$-source . . .



In **Top**:

Every $|\_|$-source has a lift



In **Top**:
Can find topology B on D such that:

- each g_i is continuous

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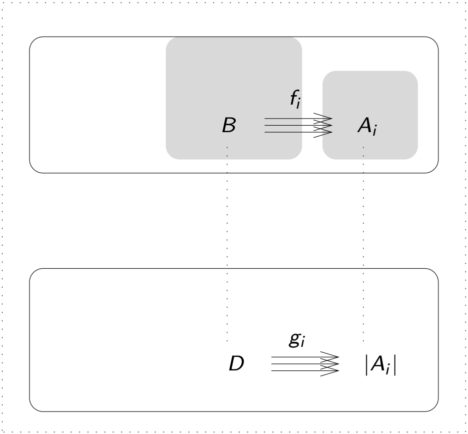

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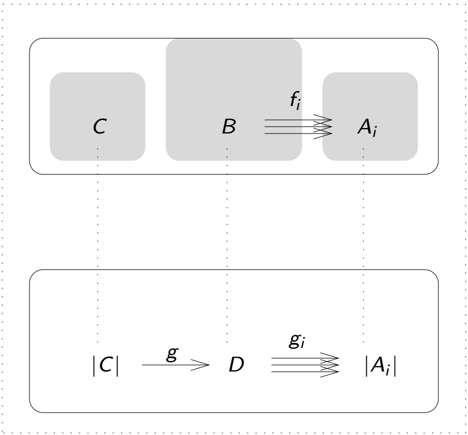

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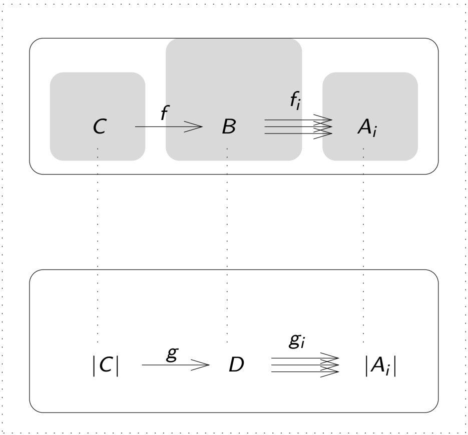

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g lifts (is continuous) iff $g_i \circ g$ all lift

Duality

Topological functors are *self-dual*.
Equivalent definition:

Definition

$|\_| : \mathbf{C} \to \mathbf{D}$ is a *topological* functor iff:

- faithful
- amnestic — "finer" is antisymmetric
- and every $|\_|$-*sink* has a $|\_|$-*final* lift — "a finest topology"

A few serendipitous facts

If $|\_|$ topological, then:

- Discrete object functor $\_{}_0 \dashv |\_| \dashv \_{}_1$ Indiscrete object functor
- preserves (co)limits
- lifts (co)limits

A few serendipitous facts, and a subtlety

If $|\_|$ topological, then:

- Discrete object functor $\_0 \dashv |\_| \dashv \_1$ Indiscrete object functor
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Definition

$|\_|$ *well-fibered* iff

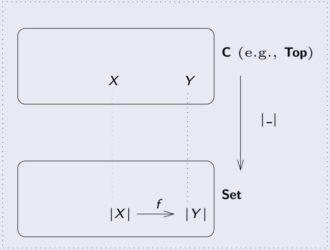
- every fiber is *small*
- $1_0 = 1_1$ "only one topology on terminal object"

The $\mathbf{Map}_{\mathcal{C}}$ Category, Categorically

Fix $|\_| : \mathbf{C} \to \mathbf{Set}$ topological and a class $\mathcal{C}$ of objects of $\mathbf{C}$

Definition ($\mathcal{C}$-map)

f $\mathcal{C}$-map
iff $f \circ |k|$ lifts for every $\mathcal{C}$-probe k.



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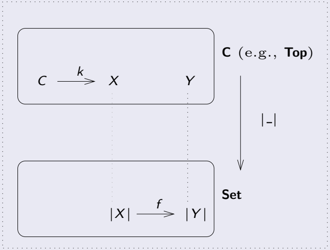

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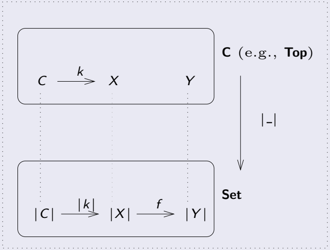

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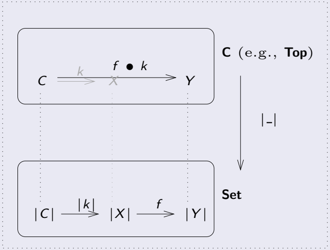

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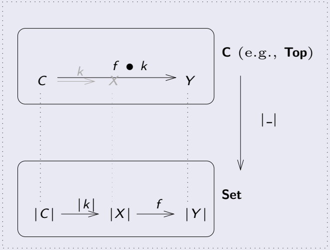


Definition (**Map**$_{\mathcal{C}}$)

Objects=those of $\mathbf{C}$. Morphisms=$\mathcal{C}$-*maps*.

Note: every morphism is a $\mathcal{C}$-map.

The $\mathbf{Map}_{\mathcal{C}}$ Category is Cartesian-closed

Theorem

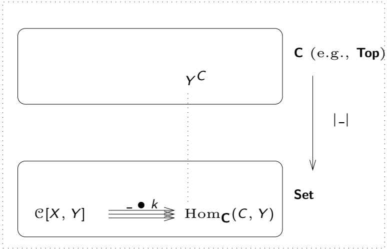
Let $|\_| : \mathbf{C} \to \mathbf{Set}$ be topological well-fibered.
If $\mathcal{C}$ is strongly productive, i.e.,

- *objects of $\mathcal{C}$ are exponentiable*
- *$\mathcal{C}$ closed under binary products*

then $\mathbf{Map}_{\mathcal{C}}$ is Cartesian-closed.

Proof. Exponential $[Y^X]_{\mathcal{C}}$ is:

- (in the fiber of) the set $\mathcal{C}[X, Y]$ of $\mathcal{C}$-maps : $X \to Y$



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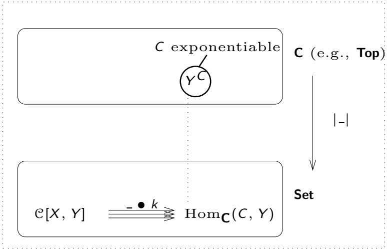

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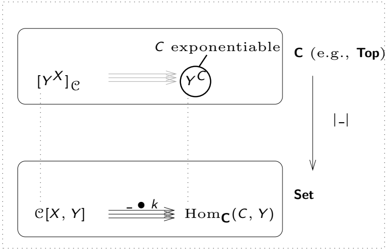
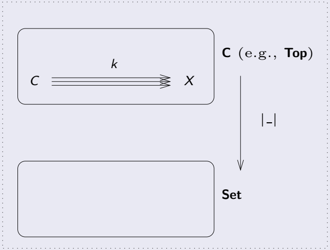


From $\mathbf{Map}_{\mathcal{C}}$ to $\mathbf{C}_{\mathcal{C}}$

- $\mathbf{Map}_{\mathcal{C}}$ *not* a subcategory of $\mathbf{C}$: *more* morphisms
- Find equivalent subcategory $\mathbf{C}_{\mathcal{C}}$ of $\mathbf{C}$ using process equivalent to k-ification.

Definition ($\mathcal{C}$-ification)

$\mathcal{C}X$ is finest in the fiber of $|X|$ such that all $\mathcal{C}$-probes $k\colon C \to X$ lift to $C \to \mathcal{C}X$



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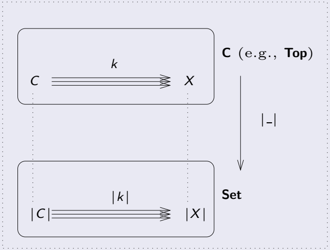

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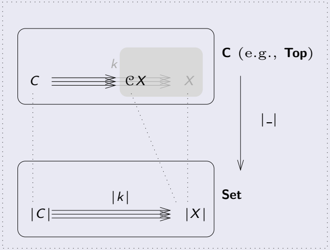


$\mathcal{C}$-generated objects

Definition

X is $\mathcal{C}$-*generated* iff $\mathcal{C}X = X$.
$\mathbf{C}_{\mathcal{C}}$ is the full subcategory of $\mathcal{C}$-generated objects in $\mathbf{C}$.

Theorem

$$\mathbf{Map}_{\mathcal{C}} \underset{|\_|}{\overset{\mathcal{C}}{\rightleftarrows}} \mathbf{C}_{\mathcal{C}}$$

is an equivalence of categories.

The Main Theorem

Theorem (Escardò-Lawson-Simpson, JGL)

Let $|\_| \colon \mathbf{C} \to \mathbf{Set}$ be topological well-fibered.
If $\mathcal{C}$ is strongly productive, then:

- $\mathbf{C}_{\mathcal{C}}$ *is coreflective in* $\mathbf{C}$
  - *Coreflection* $\mathcal{C} \colon \mathbf{C} \to \mathbf{C}_{\mathcal{C}}$
- $\mathbf{C}_{\mathcal{C}}$ *is Cartesian-closed*
  - *Product* $X \times_{\mathcal{C}} Y$ *is* $\mathcal{C}(X \times Y)$
  - *Exponential is* $\mathcal{C}([Y^X]_{\mathcal{C}})$
- $\mathbf{C}_{\mathcal{C}}$ *is cocomplete*
  - *Colimits are computed as in* $\mathbf{C}$
  - $\mathcal{C}$*-generated* $\Leftrightarrow$ *colimit of objects of* $\mathcal{C}$

Note: the proof of the latter is interesting; rests on colimit of probes $k \colon C_{X'} \to X$ such that $|k|$ does *not* lift to $C_{X'} \to X'$, for X' in fiber of $|X|$ not coarser than X.

Outline

Directed Algebraic Topology

- Originates (mostly) from computer science questions on *concurrent processes* [E. Goubault 95]
- ... interpreted geometrically [E. Dijkstra 68]

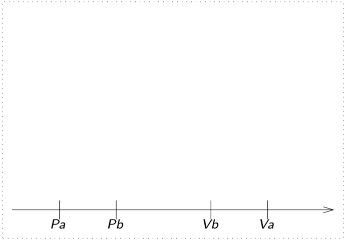
Object of study:

Topological space + notion of **time** (possibly cyclic)

PV Processes

Locks ensure access to resource by *single* process:

- Pa blocks until lock a taken
- Va releases lock a



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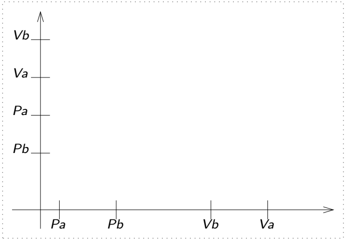

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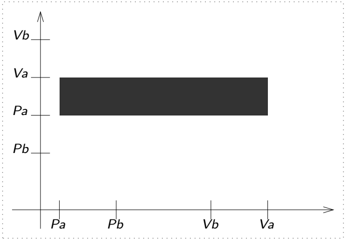

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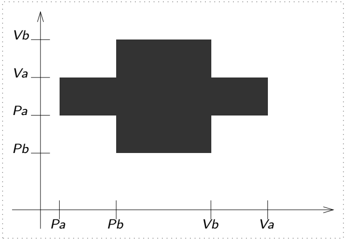

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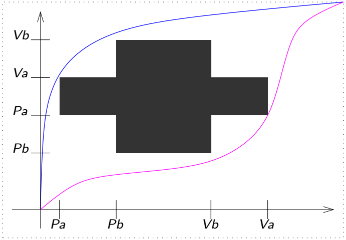

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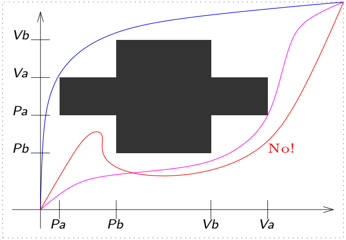

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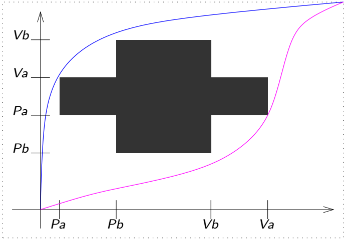

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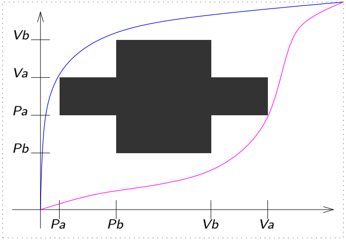

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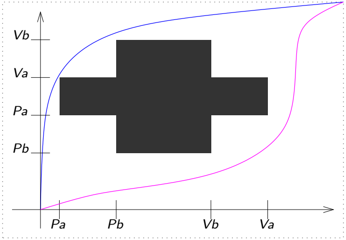

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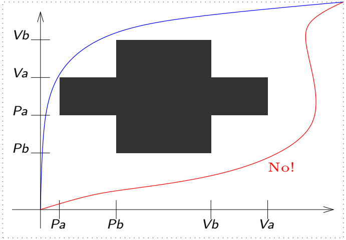

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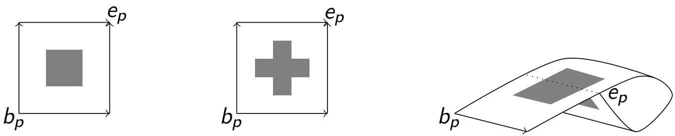

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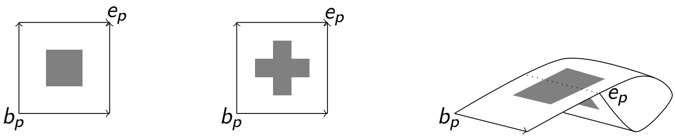

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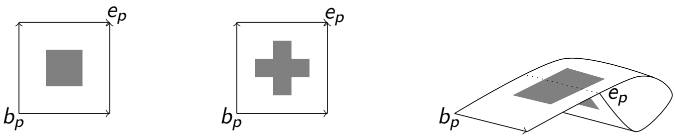

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- Better: d-spaces [Grandis 09], **streams** [Krishnan 08]

Prestreams

A prestream is a topological presheaf of preorders:

Definition (Category Prestr of prestreams)

A *prestream* $\mathcal{X}$ = topological space X
+ *precirculation* $(\sqsubseteq_U)_{U \in \mathcal{O}(X)}$:

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- $\sqsubseteq_U$ preorder on U
- monotonicity: $U \subseteq V, x \sqsubseteq_U y \Rightarrow x \sqsubseteq_V y$

Prestream *morphisms* $f : (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)}) \to (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$:

- continuous
- locally monotonic: for each open V of Y,

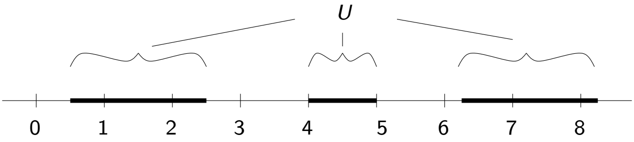
$$x \sqsubseteq_{f^{-1}(V)} y \Rightarrow f(x) \preceq_V f(y)$$

Paradigmatic examples 1

- *Preordered spaces*: $\sqsubseteq_U = \leq_{|U}$ for each U

Paradigmatic examples 1

- *Preordered spaces*: $\sqsubseteq_U = \leq_{|U}$ for each U
- $\overrightarrow{\mathbb{R}}$: $t \sqsubseteq_U^{\mathbb{R}} t'$ iff *whole interval* $[t, t'] \subseteq U$.



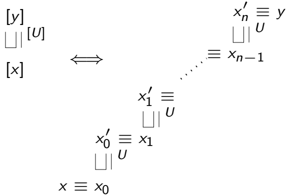
Here, $1 \sqsubseteq_U 2$, but $2 \not\sqsubseteq_U 4.6$
"Islands of order"

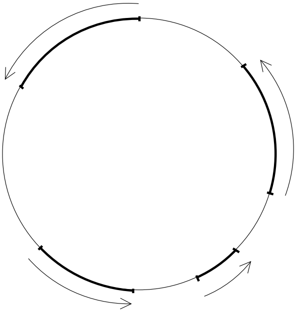
Paradigmatic examples 2: the directed circle

Definition

$$\overrightarrow{S}^1 = \overrightarrow{\mathbb{R}}/\mathbb{Z}$$

- Quotient by $x \equiv y$ iff $x - y \in \mathbb{Z}$
- **Prestr** is cocomplete, has quotients





Using the Escardò-Lawson-Simpson construction

- The forgetful functor $(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)}) \mapsto X : \textbf{Prestr} \to \textbf{Top}$ is topological
- ... hence $\textbf{Prestr}_{\mathcal{C}}$ is Cartesian-closed for any strongly productive class. Remember:
  - consisting of exponentiable objects in **Prestr**
  - closed under binary products
- What are the *exponentiable prestreams*?

Exponentiable prestreams

Theorem

The prestream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ *is exponentiable iff:*

- X *is core-compact* *(i.e., exponentiable in* **Top***)*
- $\mathcal{X}$ *is a preordered space* *(i.e.,* $\sqsubseteq_U = (\sqsubseteq_X)_{|U}$*)*

Many CCCs of prestreams

$\mathbf{Prestr}_{\mathcal{C}}$ ($\mathcal{C}$-generated prestreams):

- $\mathcal{C} =$ all preordered core-compact spaces
  $\Rightarrow$ *largest* such CCC
- $\mathcal{C} =$ preordered compact Hausdorff spaces
  $\Rightarrow$ prestream quotients of preordered loc. compact Hausdorff spaces
- $\mathcal{C} =$ compact pospaces
  $\Rightarrow$ prestream quotients of locally compact pospaces
- etc.

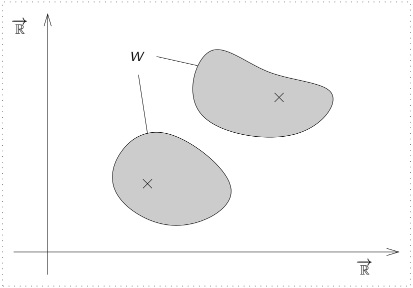
The issue with prestreams

Binary products in **Prestr** are *weird*.

Lemma

$(X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)}) \times (Y, (\preceq_V)_{V \in \mathcal{O}(Y)})$ is $X \times Y$ with precirculation:

$$(x, y) \leq_W (x', y') \text{ iff } x \sqsubseteq_{\pi_1[W]} x' \text{ and } y \preceq_{\pi_2[W]} y'$$



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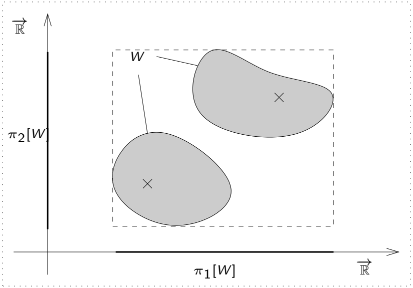

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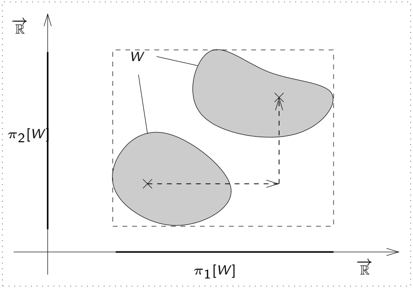
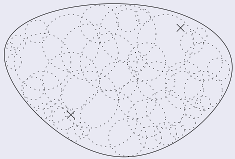


Streams

Definition (One-step cosheafification)

For $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$, $Sh^1(\mathcal{X}) = (X, (\widehat{\sqsubseteq}_U)_{U \in \mathcal{O}(X)})$ where:

$x \mathrel{\widehat{\sqsubseteq}_U} y$ iff
for every open cover $(U_i)_{i \in I}$ of U...



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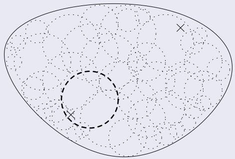

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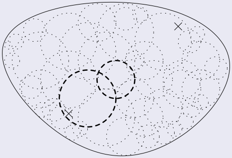

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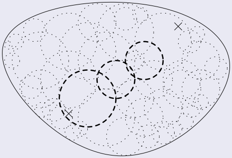

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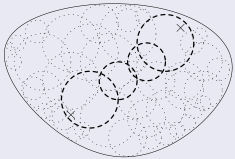

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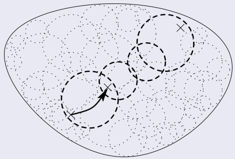

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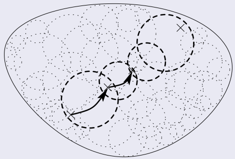

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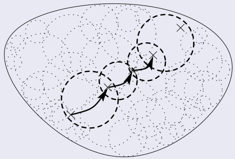

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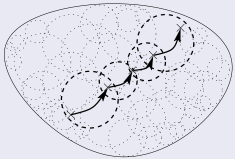


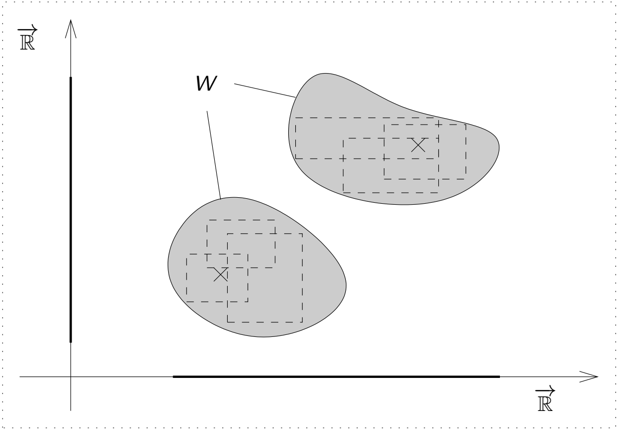
Note: $Sh^1(\mathcal{X})$ always finer than $\mathcal{X}$ on X.

Definition (Stream)

$\mathcal{X}$ stream iff $Sh^1(\mathcal{X}) = \mathcal{X}$

Defines a full subcategory **Str** of **Prestr**, topological over **Top**.

Stream products



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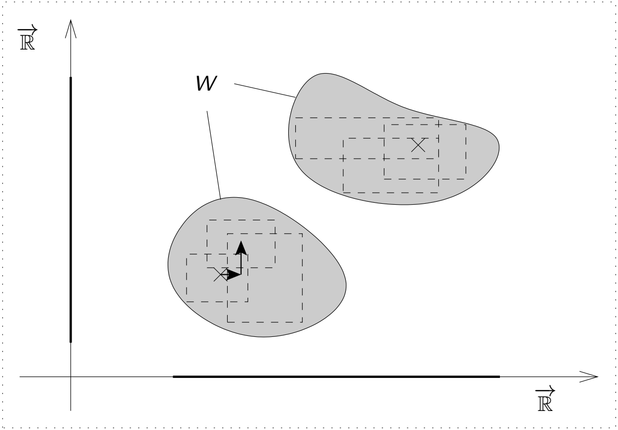

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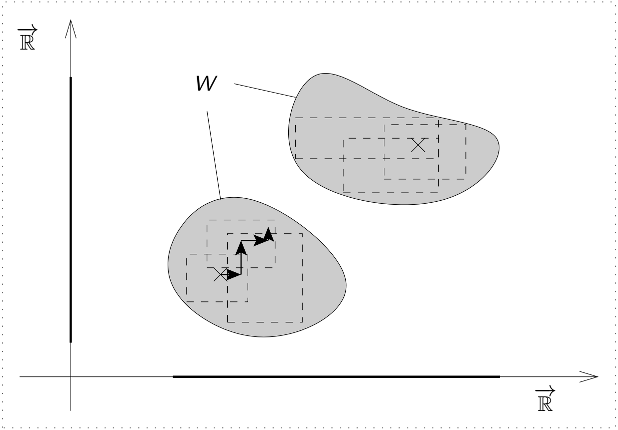

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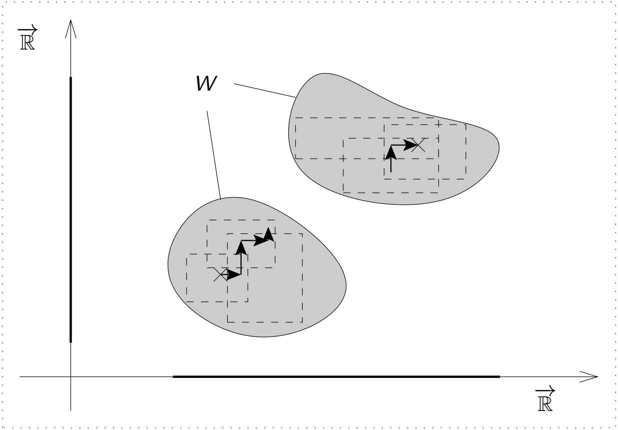

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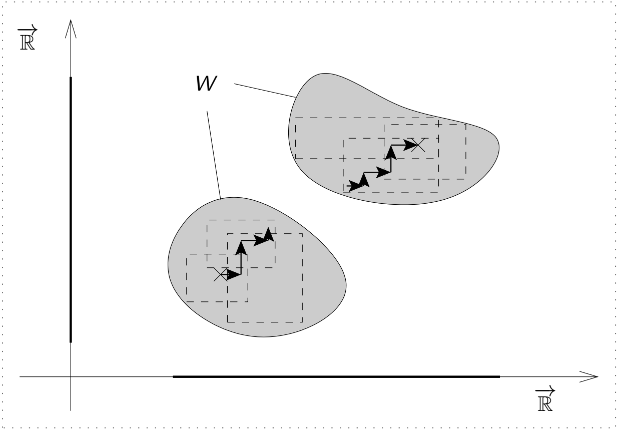
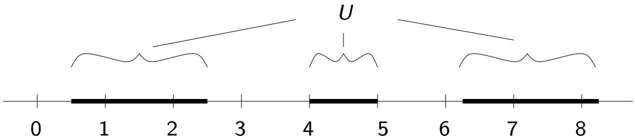
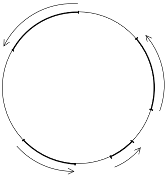


Paradigmatic examples

- *Not* preordered spaces in general, but their cosheafification OK
- $\overrightarrow{\mathbb{R}}$ = cosheafification of $(\mathbb{R}, \leq)$:



- $\overrightarrow{S}^1 = \overrightarrow{\mathbb{R}}/\mathbb{Z}$



Exponentiable prestreams

Theorem

The prestream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ *is exponentiable iff:*

- X *is core-compact* *(i.e., exponentiable in* **Top***)*
- $\mathcal{X}$ *is a preordered space*

Exponentiable ~~pre~~streams

Theorem

The ~~pre~~stream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ is exponentiable iff:

- *X is core-compact* *(i.e., exponentiable in **Top**)*
- ~~*$\mathcal{X}$ is a preordered space*~~

Exponentiable streams

Theorem

The stream $\mathcal{X} = (X, (\sqsubseteq_U)_{U \in \mathcal{O}(X)})$ *is exponentiable iff:*

- X *is core-compact* *(i.e., exponentiable in* **Top***)*

Many CCCs of streams

$\mathbf{Str}_{\mathcal{C}}$ ($\mathcal{C}$-generated prestreams):

- $\mathcal{C}$ = all core-compact streams

  $\Rightarrow$ *largest* such CCC
- $\mathcal{C}$ = compact Hausdorff streams

  $\Rightarrow$ quotients of loc. compact Hausdorff streams
- etc.

Note: the weak Hausdorff streams of the 2nd kind are Krishnan's *compactly flowing streams*.

Outline

Conclusion

- A fairly general, *simple* construction of CCCs: $\mathbf{Map}_{\mathcal{C}} \cong \mathbf{C}_{\mathcal{C}}$
- Many CCCs of topological spaces (including k-spaces)
- Many CCCs of prestreams
- Many CCCs of streams (including compactly flowing streams)

What is "Cartesian-closed"?

Definition

X is *exponentiable* if $\_ \times X$ is left adjoint.
Cartesian-closed $=$ every object is exponentiable.

Right adjoint is the *exponential* $\_^X$.

- application $\mathrm{App}\colon Y^X \times X \to Y$

$$(f, x) \mapsto f(x)$$

- currification $\Lambda(f)\colon Z \to Y^X$ for each $f\colon Z \times X \to Y$

$$z \mapsto (x \mapsto f(x, y))$$

- satisfying some equations (omitted)

Convenient in algebraic topology:

- homotopies through path functor
- geometric realization preserves finite products

Fundamental in semantics of programming languages

What about replacing topological spaces by filter spaces?

Given filter space X, let $\mathbf{U}X$ be underlying topological space.
Not checked, but seems likely:

Definition

A *(pre)fream* is X filter space + (pre)circulation $(\sqsubseteq_U)_{U \in \mathcal{O}(\mathbf{U}X)}$.

Claim

The exponentiable prefreams are the preordered filter spaces.

... hence can build CCCs of preordered-generated filter spaces, etc.

Claim

Every fream is exponentiable: the category of freams is Cartesian-closed.

Cosheafification

Fiber of X is a complete lattice.
Iterate Sh^1 transfinitely,
obtain $Sh^\infty(X)$, coarsest stream finer than $\mathcal{X}$.

Definition

$Sh^\infty(\mathcal{X})$ is the *cosheafification* of $\mathcal{X}$.

Sh^∞ is right adjoint to inclusion functor, so:

Theorem

Str *is a coreflective subcategory of* **Prestr**, *topological over* **Top**.

(General argument on categories of fixed point of deflationary endofunctors that are identity on morphisms, on categories with a fiber-small topological functor.)