Tree Automata and Applications

M1 course, 2024/2025

Mostly based on slides by Stefan Schwoon

Organization

Schedule

- Exercises: Wednesday 8:30 – 10:30 (Luc Lapointe)
- Lectures: Wednesday 10:45 – 12:45 (Laurent Doyen)

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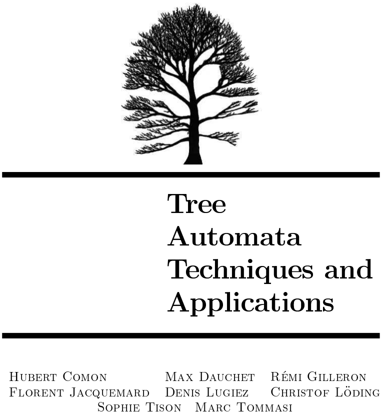
Assessment

- DM or CC (*to be specified by Luc*)
- Final Exam: 2h, 15th January 10am
- First session: DM/CC + Exam (50/50)
- Second session: DM/CC + Repeat Exam (50/50)

Material

Course material

- Website: lecturer's homepage + Wiki MPRI, course 1-18 (exercise sheets, slides, former exams)
- Main reference: H. Comon et al. Tree Automata Techniques and Applications, 2008.



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- Website: lecturer's homepage + Wiki MPRI, course 1-18 (exercise sheets, slides, former exams)
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Other relevant resources

- C. Löding, W. Thomas. Automata on finite trees. Handbook of Automata Theory (I.), pp. 235-264, 2021.
- L. Doyen. Top-Down Complementation of Automata on Finite Trees. IPL 187:106499, 2025.

Motivation

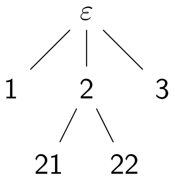
Context

1. Natural extension of formal languages and automata on words
2. Connection with Logic & Games
3. Treatment of tree-like data structures: parse trees, XML documents
4. Applications e.g. in compiler construction, formal verification

Trees

We consider *finite* *ordered* *ranked* trees. Let $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$

- *finite* set nodes (positions), denoted by $Pos \subseteq \mathbb{N}_0^*$ (with $\varepsilon \in Pos$)
- *ordered* : internal nodes have children $1, \ldots, n$
- *ranked* : number of children fixed by node's label



Definition: Tree

A (finite, ordered) *tree* is a nonempty, finite, prefix-closed set $Pos \subseteq \mathbb{N}_0^*$ such that $w \cdot (i+1) \in Pos$ implies $w \cdot i \in Pos$ for all $w \in \mathbb{N}^*$, $i \in \mathbb{N}_0$.

- In the sequel, we write wi instead of $w \cdot i$
- prefix-closed: $wi \in Pos$ implies $w \in Pos$

Ranked Trees

Ranked symbols

Ranked alphabet $\mathcal{F}$: finite set of symbols, each with an *arity* $0, 1, \ldots$
Denote by $\mathcal{F}_i$ the symbols of arity i (hence $\mathcal{F} := \bigcup_i \mathcal{F}_i$).

- arity 0: constants
- arity ≥ 1: functions (unary, binary, etc.)

Notation (example): $\mathcal{F} = \{f(2), g(1), a, b\}$
Let $\mathcal{X}$ denote a set of variables (of arity 0), disjoint from the other symbols.

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Definition: Ranked tree

A ranked tree is a mapping $t : Pos \to (\mathcal{F} \cup \mathcal{X})$ satisfying:

- Pos is a tree;
- for all $p \in Pos$, if $t(p) \in \mathcal{F}_n$, $n \geq 1$ then $Pos \cap p\mathbb{N} = \{p1, \ldots, pn\}$;
- for all $p \in Pos$, if $t(p) \in \mathcal{X} \cup \mathcal{F}_0$ then $Pos \cap p\mathbb{N} = \varnothing$.

Trees and Terms

Definition: Terms

The set of *terms* $T(\mathcal{F}, \mathcal{X})$ is the smallest set satisfying:

- $\mathcal{X} \cup \mathcal{F}_0 \subseteq T(\mathcal{F}, \mathcal{X})$;
- if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{X})$ and $f \in \mathcal{F}_n$, then $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{X})$.

We write $T(\mathcal{F}) := T(\mathcal{F}, \varnothing)$, called the set of *ground terms*.
A term of $T(\mathcal{F}, \mathcal{X})$ is *linear* if every variable occurs at most once.

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Example: $\mathcal{F} = \{f(2), g(1), a, b\}$, $\mathcal{X} = \{x, y\}$

- $f(g(a), b) \in T(\mathcal{F})$;
- $f(x, f(b, y)) \in T(\mathcal{F}, \mathcal{X})$ is linear;
- $f(x, x) \in T(\mathcal{F}, \mathcal{X})$ is non-linear.

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We use 'terms' and 'trees' interchangeably (obvious bijection).

Height and Size

Definition

Let $t \in T(\mathcal{F}, \mathcal{X})$. We denote by $\mathcal{H}(t)$ the *height*, and by $|t|$ the *size*, of t.

- if $t \in \mathcal{X}$, then $\mathcal{H}(t) := 0$ and $|t| := 0$; (for notational convenience)
- if $t \in \mathcal{F}_0$, then $\mathcal{H}(t) := 1$ and $|t| := 1$;
- if $t = f(t_1, \ldots, t_n)$, then $\mathcal{H}(t) := 1 + \max\{\mathcal{H}(t_1), \ldots, \mathcal{H}(t_n)\}$ and $|t| := 1 + |t_1| + \cdots + |t_n|$.

Subterms / subtrees

Definition: Subtree

Let $t, u \in T(\mathcal{F}, \mathcal{X})$ and p a position. Then $t_{|p} : Pos_p \to T(\mathcal{F}, \mathcal{X})$ is the ranked tree defined by

- $Pos_p := \{ q \mid pq \in Pos \}$;
- $t_{|p}(q) := t(pq)$.

Moreover, $t[u]_p$ is the tree obtained by replacing $t_{|p}$ by u in t.

$t \trianglerighteq t'$ (resp. $t \triangleright t'$) denotes that t' is a (proper) subtree of t.

Substitutions and Context

Definition: Substitution

- (Ground) substitution σ: mapping from $\mathcal{X}$ to $T(\mathcal{F}, \mathcal{X})$, resp., $T(\mathcal{F})$
- Notation: $\sigma := \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$, with $\sigma(x) := x$ for all $x \in \mathcal{X} \setminus \{x_1, \ldots, x_n\}$
- Extension to terms: for all $f \in \mathcal{F}_m$ and $t'_1, \ldots, t'_m \in T(\mathcal{F}, \mathcal{X})$
$$\sigma(f(t'_1, \ldots, t'_m)) = f(\sigma(t'_1), \ldots, \sigma(t'_m))$$
- Notation: $t\sigma$ for $\sigma(t)$

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Definition: Context

A *context* is a linear term $C \in T(\mathcal{F},\mathcal{X})$ with variables $x_1, \ldots, x_n$.
We note $C[t_1, \ldots, t_n] := C\{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$.

$\mathcal{C}^n(\mathcal{F})$ denotes the contexts with n variables and $\mathcal{C}(\mathcal{F}) := \mathcal{C}^1(\mathcal{F})$.
Let $C \in \mathcal{C}(\mathcal{F})$. We note $C^0 := x_1$ and $C^{n+1} = C^n[C]$ for $n \geq 0$.

Tree automata

Basic idea: Extension of finite automata from words to trees

Direct extension of automata theory when words seen as unary terms:

$$abc \mathrel{\widehat{=}} a(b(c(\$)))$$

Finite automaton: labels every prefix of a word with a state.

Tree automaton: labels every position/subtree of a tree with a state.

Two variants: bottom-up vs top-down labelling

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Two variants: bottom-up vs top-down labelling

Basic results (preview)

- Non-deterministic bottom-up and top-down are equally powerful
- Deterministic bottom-up equally powerful
- Deterministic top-down less powerful

Bottom-up automata

Definition: (Bottom-up tree automata)

A *(finite bottom-up) tree automaton* (NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, where:

- Q is a finite set of *states*;
- $\mathcal{F}$ a finite ranked alphabet;
- $G \subseteq Q$ are the *final states*;
- Δ is a finite set of rules of the form

$$f(q_1, \ldots, q_n) \to q$$

for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

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for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Example: $Q := \{q_0, q_1, q_f\}$, $\mathcal{F} = \{f(2), g(1), a\}$, $G := \{q_f\}$, and rules

$$a \to q_0 \quad g(q_0) \to q_1 \quad g(q_1) \to q_1 \quad f(q_1, q_1) \to q_f$$

Move relation and Recognized language

$\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$

Move relation

Let $t, t' \in T(\mathcal{F}, Q)$. We write $t \rightarrow_{\mathcal{A}} t'$ if the following are satisfied:

- $t = C[f(q_1, \ldots, q_n)]$ for some context C;
- $t' = C[q]$ for some rule $f(q_1, \ldots, q_n) \rightarrow q$ of $\mathcal{A}$.

Idea: successively reduce t to a single state, starting from the leaves.
As usual, we write $\rightarrow^*_{\mathcal{A}}$ for the transitive and reflexive closure of $\rightarrow_{\mathcal{A}}$.

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Recognized Language

- A tree t is *accepted* by $\mathcal{A}$ if $t \rightarrow^*_{\mathcal{A}} q$ for some $q \in G$.
- $\mathcal{L}(\mathcal{A})$ denotes the set of trees accepted by $\mathcal{A}$.
- L is *recognizable* if $L = \mathcal{L}(\mathcal{A})$ for some NFTA $\mathcal{A}$.

NFTA with ε-moves

Definition:

An ε-NFTA is an NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, where Δ can additionally contain rules of the form $q \to q'$, with $q, q' \in Q$.

Semantics: allow to re-label a position from q to q': $C[q] \to_{\mathcal{A}} C[q']$.

Equivalence of ε-NFTA

For every ε-NFTA $\mathcal{A}$ there exists an equivalent NFTA $\mathcal{A}'$.

Proof (sketch): construct the rules of $\mathcal{A}'$ by a saturation procedure.
Initialize $\Delta' = \Delta$ and apply:

$$\frac{f(q_1, \ldots, q_n) \to q \in \Delta' \quad q \to q' \in \Delta}{f(q_1, \ldots, q_n) \to q' \in \Delta'}$$

Deterministic, complete, and reduced NFTA

An NFTA is *deterministic* if no two rules have the same left-hand side.
An NFTA is *complete* if for every $f \in \mathcal{F}_n$ and $q_1, \dots, q_n \in Q$, there exists at least one rule $f(q_1, \dots, q_n) \to q \in \Delta$.

A state q of $\mathcal{A}$ is *accessible* if there exists a tree t s.t. $t \to_{\mathcal{A}}^{*} q$.
$\mathcal{A}$ is said to be *reduced* if all its states are accessible.

Top-down tree automata

Definition

A *top-down tree automaton* (T-NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, I, \Delta \rangle$, where $Q, \mathcal{F}$ are as in NFTA, $I \subseteq Q$ is a set of *initial states*, and Δ contains rules of the form

$$q(f) \rightarrow (q_1, \ldots, q_n)$$

for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

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for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Move relation

Let $t, t' \in T(\mathcal{F}, Q)$. We write $t \to_{\mathcal{A}} t'$ if

- $t = C[q(f(t_1, \ldots, t_n))]$ for some context C;
- $t' = C[f(q_1(t_1), \ldots, q_n(t_n))]$ for some rule $q(f) \to (q_1, \ldots, q_n)$ of $\mathcal{A}$.

t is accepted by $\mathcal{A}$ if $q(t) \to_{\mathcal{A}}^{*} t$ for some $q \in I$.

From top-down to bottom-up

Theorem (T-NFTA = NFTA)

L is recognizable by an NFTA iff it is recognizable by a T-NFTA.

Claim: L is accepted by NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ iff it is accepted by T-NFTA $\mathcal{A}' = \langle Q, \mathcal{F}, I, \Delta' \rangle$, with $I = G$ and

$$\Delta' := \{ q(f) \to (q_1, \ldots, q_n) \mid f(q_1, \ldots, q_n) \to q \in \Delta \}$$

$$\text{(and vice versa) } \Delta := \{ f(q_1, \ldots, q_n) \to q \mid q(f) \to (q_1, \ldots, q_n) \in \Delta' \}$$

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Proof: Let $t \in T(\mathcal{F})$. We show $t \to^*_{\mathcal{A}} q$ iff $q(t) \to^*_{\mathcal{A}'} t$.

- Base: $t = a$ (for some $a \in \mathcal{F}_0$)

$$t = a \to^*_{\mathcal{A}} q \iff a \to_{\Delta} q \iff q(a) \to_{\Delta'} \varepsilon \iff q(a) \to^*_{\mathcal{A}'} a$$

- Induction: $t = f(t_1, \ldots, t_n)$, hypothesis holds for $t_1, \ldots, t_n$

$$f(t_1, \ldots, t_n) \to^*_{\mathcal{A}} q \iff \exists q_1, \ldots q_n : f(q_1, \ldots, q_n) \to_{\Delta} q \wedge \forall i : t_i \to^*_{\mathcal{A}} q_i$$

$$\iff \exists q_1, \ldots, q_n : q(f) \to_{\Delta'} (q_1, \ldots, q_n) \wedge \forall i : q_i(t_i) \to^*_{\mathcal{A}'} t_i$$

$$\iff q(f(t_1, \ldots, t_n)) \to_{\mathcal{A}'} f(q_1(t_1), \ldots, q_n(t_n)) \to^*_{\mathcal{A}'} f(t_1, \ldots, t_n)$$

Run (Computation tree)

$\mathcal{A} = \langle Q, \mathcal{F}, I, \Delta \rangle$

Definition: (Run)

Let $t : Pos \rightarrow \mathcal{F}$ a ground tree. A *run* of $\mathcal{A}$ on t is a labelling $t' : Pos \rightarrow Q$ compatible with Δ, i.e.:

- for all $p \in Pos$, if $t(p) = f \in \mathcal{F}_n$, $t'(p) = q$, and $t'(pj) = q_j$ for all $pj \in Pos \cap pN$, then $f(q_1, \ldots, q_n) \rightarrow q \in \Delta$

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Recognized Language

- A run t' is initialized (or accepting) if $t'(\varepsilon) \in I$.
- A tree t is *accepted* by $\mathcal{A}$ if there exists an initialized run of $\mathcal{A}$ on t.

As usual, a DFTA has *at most* one run per tree.
A DCFTA as *exactly* one run per tree.

- Notation: $\mathcal{A}_q = \langle Q, \mathcal{F}, \{q\}, \Delta \rangle$ and $L_q(\mathcal{A}) = L(\mathcal{A}_q)$, so $L(\mathcal{A}) = \bigcup_{q \in I} L_q(\mathcal{A})$.

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Claim (subset construct.): Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ an NFTA recognizing L. The following DCFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G', \Delta' \rangle$ also recognizes L:

- $G' = \{ S \subseteq Q \mid S \cap G \neq \varnothing \}$
- for every $f \in \mathcal{F}_n$ and $S_1, \ldots, S_n \subseteq Q$, let $f(S_1, \ldots, S_n) \to S \in \Delta'$, where $S = \{ q \in Q \mid \exists q_1 \in S_1, \ldots, q_n \in S_n : f(q_1, \ldots, q_n) \to q \in \Delta \}$

Proof: For $t \in T(\mathcal{F})$, show $t \to^*_{\mathcal{A}'} \{ q \mid t \to^*_{\mathcal{A}} q \}$, by structural induction.

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DFTA with accessible states

In practice, the construction of $\mathcal{A}'$ can be restricted to accessible states:
Start with transitions $a \to S$, then saturate.

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Start with transitions $a \to S$, then saturate.

Deterministic top-down are less powerful

E.g., $L = \{f(a, b), f(b, a)\}$ can be recognized by DFTA but not by T-DFTA.

A pumping lemma for tree languages

Lemma

Let L be recognizable. Then there exists k such that for all $t \in L$ with $\mathcal{H}(t) > k$, there exist contexts $C, D \in T(\mathcal{F}, \{x\})$ and $u \in T(\mathcal{F})$ satisfying:

- D is non-trivial (i.e., not just a variable);
- $t = C[D[u]]$;
- for all $n \geq 0$, we have $C[D^n[u]] \in L$.

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- $t = C[D[u]]$;
- for all $n \geq 0$, we have $C[D^n[u]] \in L$.

Proof: Let k be the number of states of an NFTA $\mathcal{A}$ recognizing L.
In an accepting run on a tree $t \in L$, there exist two positions p, pp' ($p' \neq \varepsilon$) labelled by the same state q.
Let $C = t[x]_p$, $D = t_{|p}[x]_{p'}$, and $u = t_{|pp'}$, thus $t = C[D[u]]$.
The accepting run on t entails:

$$u \to^*_{\mathcal{A}} q, D[q] \to^*_{\mathcal{A}} q, \text{ and } C[q] \to^*_{\mathcal{A}} q_f, \text{ for some final state } q_f.$$

Therefore, $D^n[q] \to^*_{\mathcal{A}} q$ for all $n \geq 0$ (by induction, where $D^0[q] := q$) and

$$C[D^n[u]] \to^*_{\mathcal{A}} C[D^n[q]] \to^*_{\mathcal{A}} C[q] \to^*_{\mathcal{A}} q_f$$

A pumping lemma for tree languages

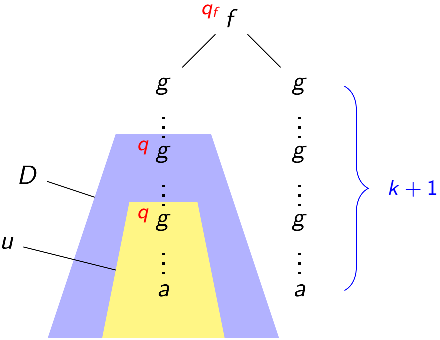
Mostly used in the form:

Lemma

If for all k,
  there exists $t \in L$ with $\mathcal{H}(t) > k$,
    for all contexts $C, D \in T(\mathcal{F}, \{x\})$ and $u \in T(\mathcal{F})$ such that
    $t = C[D[u]]$ and D is non-trivial,
      there exists $n \geq 0 : C[D^n[u]] \notin L$,
then L is not recognizable.

Illustration of pumping lemma

Let $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$ for $\mathcal{F} = \{f(2), g(1), a\}$.
Given k, let $t = f(g^k(a), g^k(a))$.



Pumping D creates trees outside L $\Rightarrow$ L not recognizable.

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Theorem (Boolean closure)

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Proof hint: uniqueness of the run on the input tree.

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Union (juxtapose)

Let $\langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ be NFTA recognizing L_i, for $i = 1, 2$.
Then $\langle Q_1 \uplus Q_2, \mathcal{F}, G_1 \cup G_2, \Delta_1 \cup \Delta_2 \rangle$ recognizes $L_1 \cup L_2$.

Cross-product construction

Direct intersection

Let $\mathcal{A}_i = \langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ be NFTA recognizing L_i, for $i = 1, 2$.
Then $\mathcal{A} = \langle Q_1 \times Q_2, \mathcal{F}, G_1 \times G_2, \Delta \rangle$ recognizes $L_1 \cap L_2$, where

$$\frac{f(q_1, \ldots, q_n) \to q \in \Delta_1 \quad f(q'_1, \ldots, q'_n) \to q' \in \Delta_2}{f(\langle q_1, q'_1 \rangle, \ldots, \langle q_n, q'_n \rangle) \to \langle q, q' \rangle \in \Delta}$$

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Remarks:

- If $\mathcal{A}_1, \mathcal{A}_2$ are D(C)FTA, then so is $\mathcal{A}$.
- If $\mathcal{A}_1, \mathcal{A}_2$ are complete, replace $G_1 \times G_2$ with $(G_1 \times Q_2) \cup (Q_1 \times G_2)$ to recognize $L_1 \cup L_2$.

Single tree

Singleton Language

Given a tree $t : Pos \to \mathcal{F}$, the language $L = \{t\}$ is recognized by $\mathcal{A}_t = \langle Q, \mathcal{F}, I, \Delta \rangle$ where:

- $Q = Pos$
- $I = \{\varepsilon\}$
- $\Delta = \{f(p1, \ldots, pn) \to p \mid f = t(p) \in \mathcal{F}_n\}$

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Remark: $\mathcal{A}_t$ is deterministic.

Proof: Show $t' \to^*_{\mathcal{A}_t} p$ iff $t' = t_{|p}$

Tree homomorphism

Definition

Let $\mathcal{X}_n := \{x_1, \ldots, x_n\}$ and $\mathcal{F}, \mathcal{F}'$ ranked alphabets.
A *tree homomorphism* is a mapping $h : \mathcal{F} \to T(\mathcal{F}', \mathcal{X})$,
with $h(f) \in T(\mathcal{F}, \mathcal{X}_n)$ if $f \in \mathcal{F}_n$.

Extension of h to trees $(T(\mathcal{F}) \to T(\mathcal{F}'))$:

- $h(f(t_1, \ldots, t_n)) = h(f)\{x_1 \leftarrow h(t_1), \ldots, x_n \leftarrow h(t_n)\}$

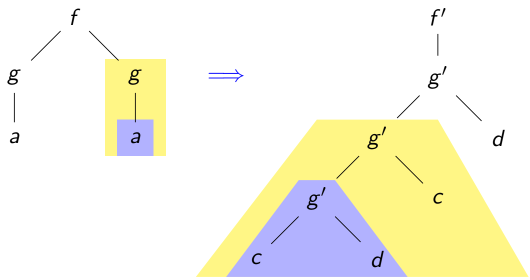
Intuition:

- $h(f)$ "explodes" f-positions into trees
- reorders/copies/deletes subtrees.

Examples

Example

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$



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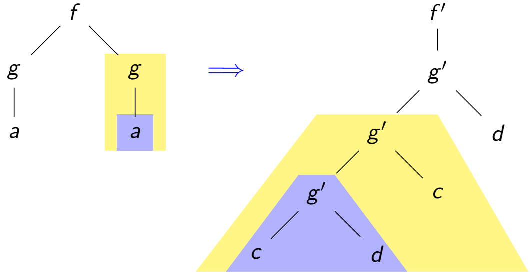

Example (ternary to binary tree)

- $\mathcal{F} = \{f(3), a, b\}$, $\mathcal{F}' = \{g(2), a, b\}$
- $h_{32}(f) = g(x_1, g(x_2, x_3))$, $h_{32}(a) = a$, $h_{32}(b) = b$

Properties of homomorphisms

A homomorphism h is

- *linear* if $h(f)$ linear for all f;
- *non-erasing* if $\mathcal{H}(h(f)) > 0$ for all f;
- *flat* if $\mathcal{H}(h(f)) = 1$ for all f;
- *complete* if $f \in \mathcal{F}_n$ implies that $h(f)$ contains all of $\mathcal{X}_n$;
- *permuting* if h is complete, linear, and flat;
- *alphabetic* if $h(f)$ has the form $g(x_1, \ldots, x_n)$ for all f.

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Non-linear homomorphisms do not preserve recognizability

- Example: $h(f) = f'(x_1, x_1)$, $h(g) = g(x_1)$, $h(a) = a$
- $L = \{ f(g^i(a)) \mid i \geq 0 \}$ (recognizable)
- $h(L) = \{ f'(g^i(a), g^i(a)) \mid i \geq 0 \}$ (not recognizable)

Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism. Then $h(L)$ is recognizable.

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- $\mathcal{A} = \langle \{q_0, q_1, q_f\}, \mathcal{F}, \{q_f\}, \Delta \rangle$ recognizes L with
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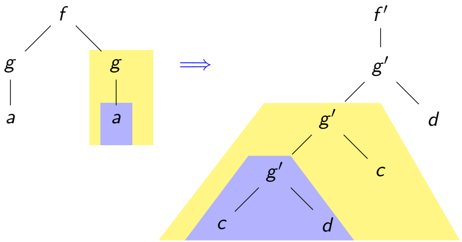

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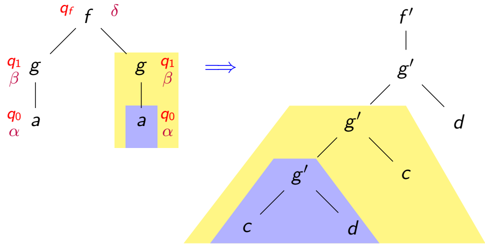
Run on $\mathcal{A}$

Rules used to produce states



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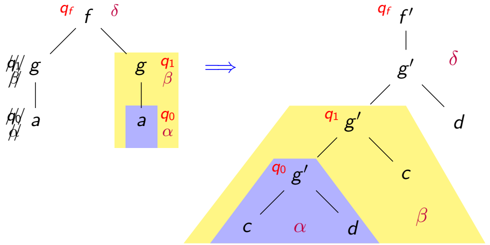
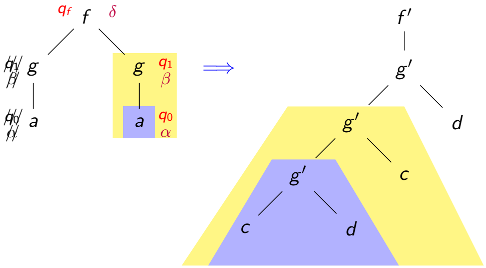


Construct automaton for $h(L)$ preserving state labels from $\mathcal{A}$ + Guess the rules.

Automaton construction for $h(L)$

Given a reduced NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ for L,
construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', G, \Delta' \rangle$ for $h(L)$.

- $Q' := Q \cup \{ \langle r, p \rangle \mid r \in \Delta, \exists f \in \mathcal{F} : r = f(\dots) \to \dots, p \in Pos_{h(f)} \}$;
- $\Delta' = \bigcup_{r \in \Delta} \Delta'_r$ where for each transition $r : f(q_1, \dots, q_n) \to q$ in Δ, the set Δ'_r contains, for all positions $p \in Pos_{h(f)}$:
  - $f'(\langle r, p1 \rangle, \dots, \langle r, pk \rangle) \to \langle r, p \rangle$ if $h(f)(p) = f' \in \mathcal{F}'_k$
  - $q_i \to \langle r, p \rangle$ if $h(f)(p) = x_i$
  - $\langle r, \varepsilon \rangle \to q$



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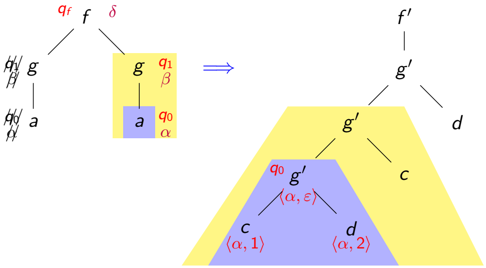

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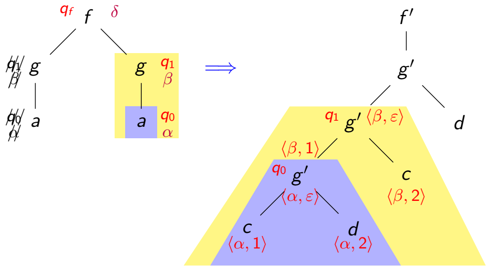

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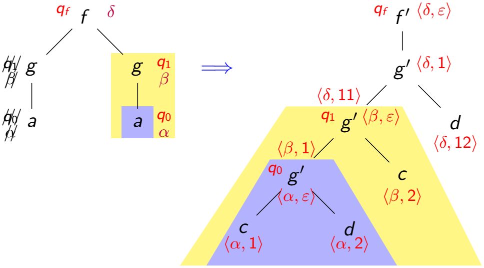

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  - Base case: $t = a$ (leaf) where $a \in \mathcal{F}_0$ (constant)
    $a \rightarrow^*_{\mathcal{A}} q \iff \underbrace{a \rightarrow q}_{r} \in \Delta$
    Then $h(a) \rightarrow^*_{\mathcal{A}'} q$ using rules in Δ'_r (single tree)
    Note: t_a is a ground term, rules $q_i \rightarrow \langle r, p \rangle$ not used

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  - Inductive case: $t = f(u_1, \ldots, u_n)$
    $t \rightarrow^*_{\mathcal{A}} \underbrace{f(q_1, \ldots, q_n) \rightarrow q}_{r \in \Delta}$ and $u_i \rightarrow^*_{\mathcal{A}} q_i$ $(i = 1, \ldots, n)$
    Then $h(t) = h(f)\{x_1 \leftarrow h(u_1), \ldots, x_n \leftarrow h(u_n)\}$
    and by induction hypothesis $h(u_i) \rightarrow^*_{\mathcal{A}'} q_i$, so $h(t) \rightarrow^*_{\mathcal{A}'} h(f)(q_1, \ldots, q_n)$
    To show: $h(f)(q_1, \ldots, q_n) \rightarrow^*_{\mathcal{A}'} q$ using rules in Δ'_r (single tree)

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- $\mathcal{L}(\mathcal{A}') \subseteq h(L)$:
  For all $t' \in T(\mathcal{F}')$, show that if $t' \rightarrow^*_{\mathcal{A}'} q \in Q$,
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  by induction on number of states (of Q) in the runs of $\mathcal{A}'$ (with ε-transitions removed) on t' corresponding to $t' \rightarrow^*_{\mathcal{A}'} q$.

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  - Base case: $t' \rightarrow^*_{\mathcal{A}'} q$, with no intermediate state from Q
    Since Δ'_r are disjoint, only rules from Δ'_r for a single r are used.
    (no variable in t' – rules $q_i \rightarrow \langle r, p \rangle$ not used)
    Let $r = f(q_1, \ldots, q_n) \rightarrow q$.
    Then $t' = h(f)$ (single tree) and we construct $t = f(u_1, \ldots, u_n)$ where $u_i \rightarrow^*_{\mathcal{A}} q_i$ (why is it possible ?).

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- $\mathcal{L}(\mathcal{A}') \subseteq h(L)$:
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  - Inductive case: $$t' \to^*_{\mathcal{A}'} v\{x_1 \leftarrow q_1, \ldots, x_n \leftarrow q_n\} \to^*_{\mathcal{A}'} \underbrace{v\{x_1 \leftarrow \langle r_1, p_1\rangle, \ldots, x_n \leftarrow \langle r_n, p_n\rangle\} \to^*_{\mathcal{A}'} q}_{\text{no intermediate state from } Q}$$
    where v is a linear term in $T(\mathcal{F}', \mathcal{X})$
    Hence $t' = v\{x_1 \leftarrow u'_1, \ldots, x_n \leftarrow u'_n\}$ where $u'_i \to^*_{\mathcal{A}'} q_i$ $(i = 1, \ldots, n)$
    (why is it possible ?)
    Also $r_1 = r_2 = \cdots = r_n = r = f(q_1, \ldots, q_n) \to q$ and $v = h(f)$ (single tree)
    By induction hyp., there exist $u_i \to^*_{\mathcal{A}} q_i$ with $h(u_i) = u'_i$ and we construct $t = f(u_1, \ldots, u_n)$ and show that $h(t) = t'$ and $t \to^*_{\mathcal{A}} q$.

Inverse tree homomorphisms

Theorem: Inverse homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F}')$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a tree homomorphism (not necessarily linear). Then $h^{-1}(L)$ is recognizable.

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Given an NFTA $\mathcal{A}' = \langle Q, \mathcal{F}', G, \Delta' \rangle$ for L,
construct NFTA $\mathcal{A} = \langle Q \uplus \{\top\}, \mathcal{F}, G, \Delta \rangle$ for $h^{-1}(L)$.

For all $n \geq 0$ and $f \in \mathcal{F}_n$, and $p_1, \ldots, p_n \in Q$,

- add $f(\top, \ldots, \top) \to \top$ to Δ;
- if $h(f)\{x_1 \leftarrow p_1, \ldots, x_n \leftarrow p_n\} \to^*_{\mathcal{A}'} q$, add $f(q_1, \ldots, q_n) \to q$ to Δ, with:

$$q_i = \begin{cases} p_i & \text{if } x_i \text{ appears in } h(f) \\ \top & \text{otherwise} \end{cases}$$

Proof: Show $t \to^*_{\mathcal{A}} q$ iff $h(t) \to^*_{\mathcal{A}'} q$, for all $t \in T(\mathcal{F})$.

Tree languages and context-free languages

Frontier

Let t be a ground tree. Then $fr(t) \in \mathcal{F}_0^*$ denotes the word obtained from reading the leaves from left to right (in increasing lexicographical order of their positions).

Example: $t = f(a, g(b, a), c)$, $fr(t) = abac$

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Leaf languages

- Let L be a recognizable tree language. Then $fr(L)$ is context-free.
- Let L be a context-free language that does not contain the empty word. Then there exists an NFTA $\mathcal{A}$ with $L = fr(\mathcal{L}(\mathcal{A}))$.

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Proof (idea):

- Given a T-NFTA recognizing L, construct a CFG from it.
- L is generated by a CFG using productions of the form $A \to BC \mid a$ only. Replace $A \to BC$ by $A \to A_2$ and $A_2 \to BC$, construct a T-NFTA from the result.

Regular Expressions

Words (alphabet Σ)

- $\emptyset, \varepsilon, a$ ($a \in \Sigma$)
- union, concatenation, iteration (Kleene star)

Trees (ranked alphabet $\mathcal{F}$)

- No empty tree
- n-ary symbols
- concatenation, iteration ?

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- No empty tree
- n-ary symbols
- concatenation, iteration ?

  $\rightsquigarrow$ use placeholders

Let $\mathcal{K} = \{\Box_1, \Box_2, \dots\}$ be a set of placeholders (symbols of arity 0).

$T(\mathcal{F}, \mathcal{K})$: set of all terms over ranked alphabet $\mathcal{F} \cup \mathcal{K}$.

Placeholders

Placeholder substitution

- Substitution: $\{\Box \leftarrow L\}$ where L is a tree language
- Can replace different occurrences of $\Box$ by different elements of L

Semantics

Based on semantics of $t\{\Box \leftarrow L\}$, by structural induction on $t \in T(\mathcal{F}, \mathcal{K})$:

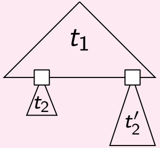
- $t = a$ of arity 0: $a\{\Box \leftarrow L\} = L$ if $a = \Box$ (otherwise $\{a\}$)
- $t = f$ of arity $n > 0$:
  $f(t_1, \ldots, t_n)\{\Box \leftarrow L\} = \{f(s_1, \ldots, s_n) \mid s_i \in t_i\{\Box \leftarrow L\}, 1 \leq i \leq n\}$
- $L_1\{\Box \leftarrow L\} = \{t\{\Box \leftarrow L\} \mid t \in L_1\}$

Abbreviation: $\{\Box_1 \leftarrow L_1, \ldots, \Box_n \leftarrow L_n\} = \{\Box_1 \leftarrow L_1\} \circ \cdots \circ \{\Box_n \leftarrow L_n\}$ $(L_i \subseteq T(\mathcal{F}))$

Concatenation - Iteration

Concatenation

$L_1 \cdot_\square L_2 = \bigcup_{t \in L_1} t\{\square \leftarrow L_2\}$



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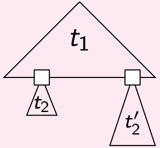

$t_1 \in L_1$
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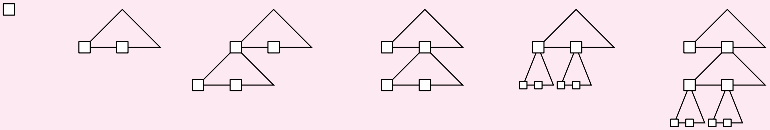
Iteration

$L^{*_\square} = \bigcup_{k\in\mathbb{N}} L^k$ where $L^0 = \{\square\}$ and $L^{k+1} = L^k \cdot_\square (L \cup L^0)$

Note: $\square \in L^{*_\square}$ for all L.

Kleene plus: $L^{+_\square} = \bigcup_{k>0} L^k$.

For $L = \{f(\square, \square, a)\}$:



Regular Expressions

Syntax

A regular expression is obtained by the following grammar:

$$E := \varnothing \mid f \mid E_1 + E_2 \mid E_1 \cdot_{\Box} E_2 \mid E^{*_{\Box}}$$

$f \in \mathcal{F}, \Box \in \mathcal{K}$

Semantics

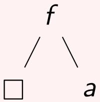
- $[\![\varnothing]\!] = \varnothing$
- $[\![f]\!] = \{f(\Box_1, \ldots, \Box_n)\}$ $(f \in \mathcal{F}_n)$
- $[\![E_1 + E_2]\!] = [\![E_1]\!] \cup [\![E_2]\!]$
- $[\![E_1 \cdot_{\Box} E_2]\!] = [\![E_1]\!] \cdot_{\Box} [\![E_2]\!]$
- $[\![E^{*_{\Box}}]\!] = [\![E]\!]^{*_{\Box}}$

A tree language L is regular if $L = [\![E]\!]$ for some regular expression E.

Shortcut: $f(E_1, \ldots, E_n) = f \cdot_{\Box_1} E_1 \ldots \cdot_{\Box_i} E_i \ldots \cdot_{\Box_n} E_n$

Regular Expressions - Examples

$E = f(\square, a)^{*\square} \cdot_{\square} a$



Regular Expressions - Examples

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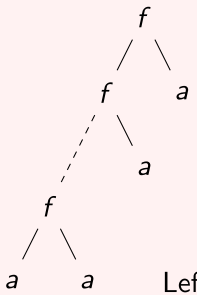

Left "comb"

Regular Expressions - Examples

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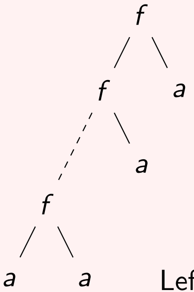
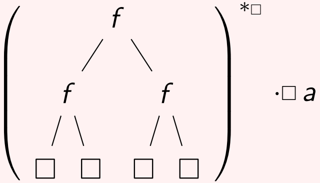


Left "comb"

$E = f(f(\square, \square), f(\square, \square))^{*\square} \cdot_\square a$



Regular Expressions - Examples

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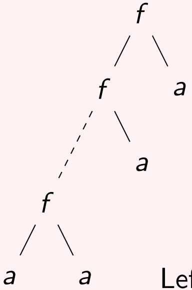
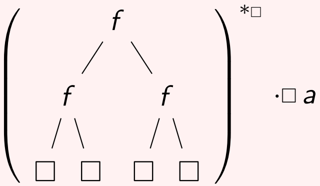


Left "comb"

$E = f(f(\square,\square), f(\square,\square))^{*_\square} \cdot_\square a$



All branches are of even length

Kleene Theorem for Tree Languages

Theorem (RegExp $\equiv$ NFTA)

L is regular iff L is recognizable by an NFTA.

Proof ($\Rightarrow$)

By induction over the structure of regular expressions:

- Base case: $\varnothing$ and $\{f(\square_1, \ldots, \square_n)\}$ are finite languages, hence recognizable.
- Inductive case: given $\mathcal{A}_i = \langle Q_i, \mathcal{F} \cup \mathcal{K}, G_i, \Delta_i \rangle$ NFTAs recognizing $L(\mathcal{A}_i) = [\![E_i]\!]$ $(i = 1, 2)$,
  - $[\![E_1 + E_2]\!]$ is recognized by $\langle Q_1 \uplus Q_2, \mathcal{F} \cup \mathcal{K}, G_1 \cup G_2, \Delta_1 \cup \Delta_2 \rangle$
  - $[\![E_1 \cdot_\square E_2]\!]$ is recognized by $\langle Q_1 \uplus Q_2, \mathcal{F} \cup \mathcal{K}, G_1, \Delta \rangle$ where $\Delta = \Delta_1 \setminus \{\square \to q \mid q \in Q_1\} \cup \Delta_2 \cup \{f(q_1, \ldots, q_n) \to q \mid \exists q' \in G_2 : f(q_1, \ldots, q_n) \to q' \in \Delta_2 \text{ and } \square \to q \in \Delta_1\}$
  - $[\![E_1^{+\square}]\!]$ is recognized by $\langle Q_1, \mathcal{F} \cup \mathcal{K}, G_1, \Delta \rangle$ where $\Delta = \Delta_1 \cup \{f(q_1, \ldots, q_n) \to q \mid \exists q' \in G_1 : f(q_1, \ldots, q_n) \to q' \in \Delta_1 \text{ and } \square \to q \in \Delta_1\}$

Kleene Theorem for Tree Languages

Theorem (RegExp $\equiv$ NFTA)

L is regular iff L is recognizable by an NFTA.

Proof ($\Leftarrow$)

Given NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, we construct a regular expression E such that $[\![E]\!] = L(\mathcal{A})$.

- Let $\mathcal{K} = \{\Box_q \mid q \in Q\}$ be a set of placeholders.
- Let $\mathcal{A}' = \langle Q, \mathcal{F} \cup \mathcal{K}, G, \Delta' \rangle$ where $\Delta' = \Delta \cup \{\Box_q \to q \mid q \in Q\}$. Then $L(\mathcal{A}') \cap T(\mathcal{F}) = L(\mathcal{A})$.
- For $q \in Q$ and $N, K \subseteq Q$, let $L(q, N, K)$ be the set of all trees $t \in T(\mathcal{F} \cup \{\Box_q \mid q \in K\})$ having a run r of $\mathcal{A}'$ such that:
  - $r(\varepsilon) = q$
  - $r(p) \in N$ for all positions $p \neq \varepsilon$ such that $t(p) \in \mathcal{F}$
  - (note: the leaves of t are labeled by $\mathcal{F}_0 \cup \{\Box_q \mid q \in K\}$)

Kleene Theorem for Tree Languages

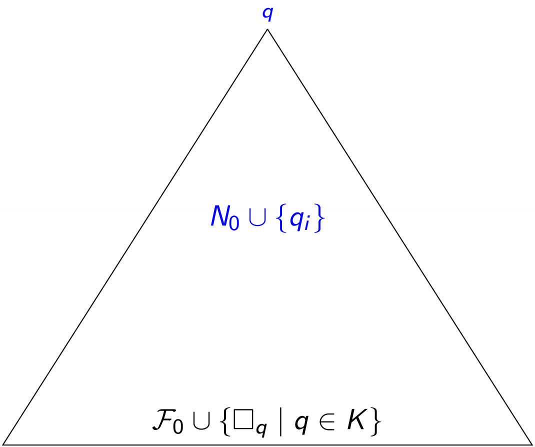
Proof ($\Leftarrow$) (cont'd)

Since $L(\mathcal{A}) = \bigcup_{q \in G} L(q, Q, \varnothing)$, showing that all sets $L(q, N, K)$ are regular is sufficient. By induction on $|N|$:

- Base case: $N = \varnothing$, then all languages $L(q, \varnothing, K)$ are finite (trees of height at most 1), hence regular.
- Inductive case: let $N = N_0 \cup \{q_i\}$ $(q_i \notin N_0)$,
  Given a run r on $t \in L(q, N, K)$, decompose t into subtrees with:
  (1) root labeled by q_i in r (by q in topmost subtree),
  (2a) internal nodes are either labeled by states of N_0 in r,
  (2b) or labeled labeled by q_i in r (and their t-symbol is replaced by $\Box_{q_i}$).

  By (2) and induction hyp., we can construct a regular exp. for the subtree-components. We construct a reg. exp. for $L(q, N, K)$ using:

  $$L(q, N_0 \cup \{q_i\}, K) = L(q, N_0, K) +$$
  $$L(q, N_0, K \cup \{\Box_{q_i}\}) \cdot_{\Box_{q_i}} (L(q_i, N_0, K \cup \{\Box_{q_i}\}))^{*_{\Box_{q_i}}} \cdot_{\Box_{q_i}} L(q_i, N_0, K)$$



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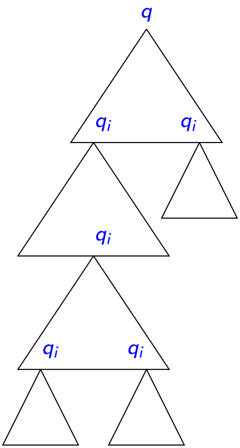

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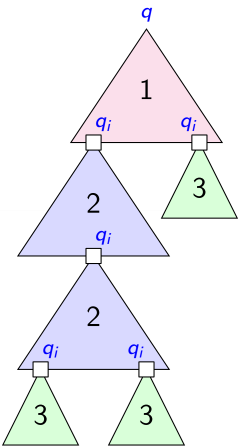

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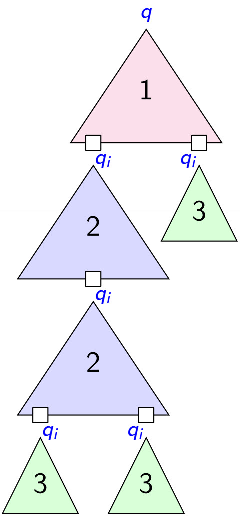
$$\underbrace{L(q, N_0, K \cup \{\Box_{q_i}\})}_{1} \cdot_{\Box_{q_i}} \big(\underbrace{L(q_i, N_0, K \cup \{\Box_{q_i}\})}_{2}\big)^{*_{\Box_{q_i}}} \cdot_{\Box_{q_i}} \underbrace{L(q_i, N_0, K)}_{3}$$



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Congruences on trees

Definition: Congruence

Let $\equiv$ be an equivalence relation on $T(\mathcal{F})$.

- $\equiv$ is called a *congruence*
  if for all $n \geq 0$ and $f \in \mathcal{F}_n$, $u_1 \equiv v_1, \ldots, u_n \equiv v_n$ we have
  $$f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n)$$
- $\equiv$ *saturates* L if $u \equiv v$ implies $u \in L \iff v \in L$.

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For $L \subseteq T(\mathcal{F})$, write $u \equiv_L v$ if

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Myhill-Nerode Theorem for trees

The following are equivalent:

1. $L \subseteq T(\mathcal{F})$ is recognizable.
2. L is saturated by some congruence of finite index.
3. $\equiv_L$ is of finite index.

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

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Proof of the theorem (sketch):

- $1 \to 2$: Let $\mathcal{A}$ be DCFTA and let $u \equiv v$ iff $u \to^*_{\mathcal{A}} q \;{}^*_{\mathcal{A}}\!\leftarrow v$.
  Then $\equiv$ is of finite index and saturates L.

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- $3 \to 1$: Let $\mathcal{A} = \langle T(\mathcal{F})_{/\equiv_L}, \mathcal{F}, L_{/\equiv_L}, \Delta \rangle$, with Δ containing

  $$f([u_1], \ldots, [u_n]) \to [f(u_1, \ldots, u_n)]$$

  for all $n \geq 0$, $f \in \mathcal{F}_n$, $u_1, \ldots, u_n \in T(\mathcal{F})$,
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Remark: This can be shown to be the canonical minimal DCFTA.

Decision Problems (on words)

(*) $\mathcal{A}, \mathcal{B}$: nondeterministic automata

- **Emptiness**

  Given $\mathcal{A}$, is $L(\mathcal{A}) = \varnothing$?

- **Universality**

  Given $\mathcal{A}$, is $L(\mathcal{A}) = \Sigma^*$?

- **Language Inclusion**

  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) \subseteq L(\mathcal{B})$?

- **Language Equivalence**

  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$?

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  Given $\mathcal{A}$, is $L(\mathcal{A}) = \varnothing$?

- **Universality**

  Given $\mathcal{A}$, is $L(\mathcal{A}) = T(\mathcal{F})$?

- **Language Inclusion**

  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) \subseteq L(\mathcal{B})$?

- **Language Equivalence**

  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$?

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(*) $\mathcal{A}, \mathcal{B}$: nondeterministic automata

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- **Language Inclusion**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) \subseteq L(\mathcal{B})$? EXPTIME-complete
- **Language Equivalence**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$? EXPTIME-complete

Emptiness of NFTA

Emptiness is P-complete

- in P: reachable states by (bottom-up) saturation algorithm
- P-hard: reduction from AND-OR graph reachability

AND-OR graph: $G = \langle V_A \uplus V_O, E \rangle$

We say that v_t is *reachable* from u in G if

- $u = v_t$, or
- $u \in V_A$, and v_t is reachable from v for all $v \in E(u)$, or
- $u \in V_O$, and v_t is reachable from v for some $v \in E(u)$

AND-OR graph reachability: given $\langle G, v_s, v_t \rangle$, is v_t reachable from v_s ?

Reduction: T-NFTA $\mathcal{A} = \langle V_A \cup V_O, \mathcal{F}, \{v_s\}, \Delta \rangle$ where Δ contains:

- $v_t(a) \rightarrow \varepsilon$,
- $u(f_n) \rightarrow (v_1, \ldots, v_n)$ for all $u \in V_A$ where $E(u) = \{v_1, \ldots, v_n\}$,
- $u(f_1) \rightarrow v$ for all $u \in V_O$, $v \in E(u)$.

Reminder: NFA universality

NFA universality is PSPACE-complete

- in (N)PSPACE: emptiness of subset construction
- PSPACE-hard: reduction from membership problem of PSpace TM

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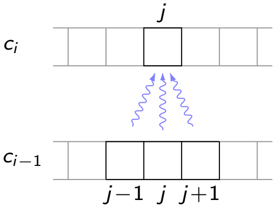
Execution of a TM on input word w:

$$c_0 \vdash c_1 \vdash \cdots \vdash c_k$$

where $c_i \in \Sigma^*$ with $\Sigma = \Gamma \cup (Q \times \Gamma)$
$c_0 = (q, w_0)w_1 w_2 \ldots w_n$, accepts if $c_k \in (Q_{acc} \times \Gamma)\Gamma^*$.

The successor relation $\vdash$ is determined by a function $\text{Next} : \Sigma^3 \to \Sigma$:
$c_{i+1,j} = \text{Next}(c_{i,j-1}, c_{i,j}, c_{i,j+1})$



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Given $\mathcal{M}$ with space bounded by $p(\cdot)$ and input word w, construct $\mathcal{A}_\mathcal{M}$ to accept (encoding) of accepting runs of $\mathcal{M}$ on w:

- $\mathcal{A}_\mathcal{M}$ has alphabet Σ
- $\mathcal{A}_\mathcal{M} = \mathcal{A}_{init} \cap \bigcap_{1 \leq i \leq p(|w|)} \mathcal{A}_i \cap \mathcal{A}_{final}$ (interesection of DFAs)
  - $L(\mathcal{A}_{init})$: run starts with c_0
  - $L(\mathcal{A}_i)$: the i-th tape cell is correctly updated along the run
  - $L(\mathcal{A}_{final})$: run contains some $q \in Q_{acc}$

How to proceed deterministically?

How many states in $\mathcal{A}_{init}$? in $\mathcal{A}_i$? in $\mathcal{A}_{final}$?

- $L(\mathcal{A}_\mathcal{M}) \neq \emptyset$ iff $\mathcal{M}$ accepts w.

Let $\mathcal{A} = \overline{\mathcal{A}}_{init} \cup \bigcup_{1 \leq i \leq p(|w|)} \overline{\mathcal{A}}_i \cup \overline{\mathcal{A}}_{final}$

Then $L(\mathcal{A}) \neq \Sigma^*$ iff $\mathcal{M}$ accepts w.

How many states in $\mathcal{A}$?

Decision Problems (on words)

(*) $\mathcal{A}, \mathcal{B}$: nondeterministic automata

- **Emptiness**
  Given $\mathcal{A}$, is $L(\mathcal{A}) = \varnothing$?
  NL-complete
- **Universality**
  Given $\mathcal{A}$, is $L(\mathcal{A}) = \Sigma^*$?
  PSPACE-complete
- **Language Inclusion**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) \subseteq L(\mathcal{B})$?
  PSPACE-complete
- **Language Equivalence**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$?
  PSPACE-complete
- **Intersection Emptiness**
  Given DFA $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\bigcap_i L(\mathcal{A}_i) = \varnothing$?
  PSPACE-complete

Decision Problems (on trees)

(*) $\mathcal{A}, \mathcal{B}$: nondeterministic automata

- **Emptiness**
  Given $\mathcal{A}$, is $L(\mathcal{A}) = \varnothing$? — P-complete

- **Universality**
  Given $\mathcal{A}$, is $L(\mathcal{A}) = T(\mathcal{F})$? — EXPTIME-complete

- **Language Inclusion**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) \subseteq L(\mathcal{B})$? — EXPTIME-complete

- **Language Equivalence**
  Given $\mathcal{A}, \mathcal{B}$, is $L(\mathcal{A}) = L(\mathcal{B})$? — EXPTIME-complete

- **Intersection Emptiness**
  Given NFTA $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\bigcap_i L(\mathcal{A}_i) = \varnothing$? — EXPTIME-complete

  (even top-down or bottom-up DFTA)

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \varnothing$?

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Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \varnothing$?

Proof (sketch):

- in EXPTIME: compute reachable tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.
- Hardness: reduction from membership problem of alternating TM with polynomial space.
  Runs of ATM are encoded as trees.
  Construct a product of tree automata to recognize accepting runs of the ATM on input word:
  - the run starts with c_0 ($\mathcal{A}_{init}$)
  - the i-th tape cell is correctly updated along all branches ($\mathcal{A}_i$)
  - all branches contain some $q \in Q_{acc}$ ($\mathcal{A}_{final}$)

Can we proceed (top-down/bottom-up) deterministically?

Path languages

Path languages

Let $t \in T(\mathcal{F})$. The *path language* $\pi(t)$ is defined as follows:

- if $t = a \in \mathcal{F}_0$, then $\pi(t) = \{a\}$;
- if $t = f(t_1, \ldots, t_n)$, for $f \in \mathcal{F}_n$, then $\pi(t) = \{\, fiw \mid w \in \pi(t_i) \,\}$.

We write $\pi(L) = \bigcup \{\, \pi(t) \mid t \in L \,\}$ for $L \subseteq T(\mathcal{F})$.

Example: $L = \{f(a, b), f(b, a)\}$, $\pi(L) = \{f1a, f2b, f1b, f2a\}$.

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Path closure

Let $L \subseteq T(\mathcal{F})$ be a tree language.

- The *path closure* of L is $pc(L) = \{\, t \mid \pi(t) \subseteq \pi(L) \,\} \supseteq L$.
- L is called *path-closed* if $L = pc(L)$.

Example: $pc(L) = \{f(a, a), f(a, b), f(b, a), f(b, b)\}$, so L is not path-closed.

Path closure and T-NFTA

Lemma

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language. Then:

- $\pi(L)$ is a recognizable word language.
- $pc(L)$ is a recognizable tree language.

Path closure and T-NFTA

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Proof: Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.

- Construct a finite (word) automaton out of $\mathcal{A}$.
  (Easy, but does require $\mathcal{A}$ to be reduced!)

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- Construct $\mathcal{A}^{\text{pc}} = \langle Q, \mathcal{F}, G, \Delta' \rangle$ for $pc(L)$ as follows:
  for all $a \in \mathcal{F}_0$:

$$q(a) \rightarrow_{\Delta} \varepsilon \quad \rightarrow \quad q(a) \rightarrow_{\Delta'} \varepsilon$$

  for all $n \geq 1$, $f \in \mathcal{F}_n$:

$$\begin{array}{l} q(f) \rightarrow_{\Delta} (q_{i,1}, \ldots, q_{i,n}) \\ i = 1, \ldots, n \end{array} \quad \rightarrow \quad q(f) \rightarrow_{\Delta'} (q_{1,1}, \ldots, q_{n,n})$$

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  $$\begin{array}{l} q(f) \rightarrow_{\Delta} (q_{i,1}, \ldots, q_{i,n}) \\ i = 1, \ldots, n \end{array} \quad \rightarrow \quad q(f) \rightarrow_{\Delta'} (q_{1,1}, \ldots, q_{n,n})$$
  Show $L_q(\mathcal{A}^{\text{pc}}) = pc(L_q(\mathcal{A}))$, i.e., $t \in L_q(\mathcal{A}^{\text{pc}}) \Leftrightarrow \pi(t) \subseteq \pi(L_q(\mathcal{A}))$
  for all $q \in Q$, $t \in T(\mathcal{F})$ (by induction).

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Path closure and T-NFTA

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Theorem

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
L is path-closed iff it is recognized by a T-DFTA.

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Theorem

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
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Proof:

- "$\rightarrow$": Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.
  Construct a T-DFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, \{G\}, \Delta' \rangle$ as follows:
  - for $a \in \mathcal{F}_0$, let $S(a) \rightarrow_{\Delta'} \varepsilon$ if $\exists q \in S, q(a) \rightarrow_{\Delta} \varepsilon$;
  - for $f \in \mathcal{F}_n$ $(n \geq 1)$, let $S(f) \rightarrow_{\Delta'} (S_1, \ldots, S_n)$
    where $S_i = \{ q_i \mid \exists q \in S, q(f) \rightarrow_{\Delta} (q_1, \ldots, q_n) \}$.

  Show that $L_S(\mathcal{A}') = \bigcup_{q \in S} L_q(\mathcal{A})$, for all $S \subseteq Q$.

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  Show that $L_S(\mathcal{A}') = \bigcup_{q \in S} L_q(\mathcal{A})$, for all $S \subseteq Q$.
- "$\leftarrow$":
  Let $\mathcal{A}$ be a redcued T-DCFTA for L. Prove that
  if $\pi(t) \subseteq \pi(L_q(\mathcal{A}))$, then $t \in L_q(\mathcal{A})$, for all $q \in Q, t \in T(\mathcal{F})$.