Verification and Game Theory

Tutorial on Basic Game Theory

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my co-authors Nicolas Markey, Romain Brenguier,
Michael Ummels, Nathan Thomasset
Stéphane Le Roux for recent discussions on the subject
Thomas Brihaye for some of the slides

The tutorial in perspective

General objective of the research topic

- Import game theory solutions to the verification field
- Lift reasoning based on two-player zero-sum games to multiplayer games

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two-player zero-sum games	multiplayer non-zero-sum games
winning objective	payoff function
winning strategy	equilibria (various kinds)
von Neumann Theorem	Nash Theorem
...	...

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Focus of the tutorial

- Give basics of game theory
- Discuss aspects that will be helpful for analyzing models useful for verification

Outline

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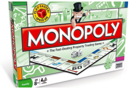
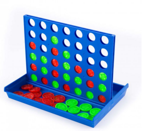
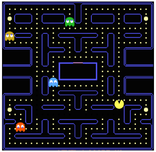
Games we play for fun







These games can be classified

- Number of players: 1 or 2 or 3 or . . .

1	$\rightsquigarrow$	*Pacman, Candy Crush, Freecel...*
2	$\rightsquigarrow$	*Chess, Tennis, Stratego, Four in a row, ...*
3 (or more)	$\rightsquigarrow$	*Poker, Monopoly,...*

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- Number of players: 1 or 2 or 3 or . . .
- Type of interactions: simultaneous or sequential

  simultaneous $\rightsquigarrow$ *Rock-Paper-Scissor, Penalty,...*
  sequential $\rightsquigarrow$ *Chess, Stratego, ...*

These games can be classified

- Number of players: 1 or 2 or 3 or . . .

- Type of interactions: simultaneous or sequential

- Maximal length of a play: finite ou infinite

  finite $\rightsquigarrow$ *Four in a row, Battleship,...*
  infinite $\rightsquigarrow$ *Tennis, Monopoly,...*

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  perfect $\rightsquigarrow$ *Four in a row, Chess,...*
  imperfect $\rightsquigarrow$ *Battleship, Poker, Stratego...*

These games can be classified

- Number of players: 1 or 2 or 3 or ...
- Type of interactions: simultaneous or sequential
- Maximal length of a play: finite ou infinite
- Type of information: perfect or imperfect
- Presence of randomness: deterministic or probabilistic

  deterministic $\rightsquigarrow$ *Four in a row, Chess, Battleship,...*
  probabilistic $\rightsquigarrow$ *Monopoly, Poker,...*

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  boolean $\rightsquigarrow$ *Four in a row, Chess,...*

  quantitative $\rightsquigarrow$ *Poker,...*

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A broader sense: What is game theory?

Goal: Model and analyze (using mathematical tools) situations of interactive decision making

[MSZ13] Maschler, Solan, Zamir. Game theory *(Cambridge University Press)*

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Wide range of applicability

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"[...] it is a context-free mathematical toolbox"

- Social science: e.g. social choice theory
- Theoretical economics: e.g. models of markets, auctions
- Political science: e.g. fair division
- Biology: e.g. evolutionary biology
- ...

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The prisoner dilemma

Two suspects are arrested by the police. The police, having separated both prisoners, visit each of them to offer the same deal.

- If one testifies (`Defects`) for the prosecution against the other and the other remains silent (`Cooperates`), the betrayer goes free and the silent accomplice receives the full 10-year sentence.
- If both remain silent, both are sentenced to only 3 years in jail.
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Modelled as a matrix game

	C	D
C	$(-3,-3)$	$(-10,0)$
D	$(0,-10)$	$(-5,-5)$

Cournot competition

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What should be the amount of the output to optimise the profit?

Selling ice-cream on the beach...

Consider a beach that can be represented by a unit interval. Sun-tanned people are located uniformly on the beach. Everyone at the beach dreams of an ice-cream.

Two ice-cream sellers will settle on the beach.

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Two ice-cream sellers will settle on the beach.

Where should they build their stand in order to optimise their benefits ?

The Nim game

The rules (simplified version)

- Two players, turn-based games
- Initially, there are 8 matches
- On each turn, a player must remove 1 or 2 matches
- The player removing the last match wins the game

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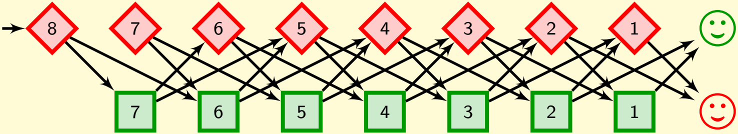
Modelled as a game played on a graph



Various models of games

Many models of games

- Strategic games
- Repeated games
- Games played on graphs
- Games played using equations
- ...

Many features

- imperfect information
- presence of randomness
- continuous time
- ...

Let us suppose that:

- we have fixed a game,
- we have identified an adequate model for this game.

The next natural question is:

What is a **solution** for this game?

Outline

Strategic games (aka matrix games, or one-shot games)

Strategic game

A strategic game G is a triple $\left(\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}}\right)$ where:

- Agt is the finite and non empty set of players,
- Σ is a non empty set of actions,
- $g_A : \Sigma^{\mathsf{Agt}} \to \mathbb{R}$ is the payoff function of player $A \in \mathsf{Agt}$.

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Example: Prisoner dilemma

- $\mathsf{Agt} = \{A_1, A_2\}$,
- $\Sigma = \{\mathtt{C}, \mathtt{D}\}$
- (g_{A_1}, g_{A_2}) is given by

	C	D
C	$(-3, -3)$	$(-10, 0)$
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Hypotheses made in classical game theory

Hypotheses

- The players are **intelligent** (i.e. they reason perfectly and quickly)
- The players are **rational** (i.e. they want to maximise their payoff)
- The players are **selfish** (i.e. they only care for their own payoff)

Outline

Optimality

Dominating profile

A profile $\mathbf{b} \in \Sigma^{\mathsf{Agt}}$ is dominating if

$$\forall \mathbf{c} \in \Sigma^{\mathsf{Agt}}\ \forall A \in \mathsf{Agt} \qquad g_A(\mathbf{c}) \leq g_A(\mathbf{b})$$

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	L	R
T	$(0, 0)$	$(2, 1)$
B	$(3, 2)$	$(1, 2)$

- $(\mathtt{B}, \mathtt{L})$ is optimal!

Strict domination

Stricly dominated action (or strategy)

An action $b_A \in \Sigma$ is strictly dominated by $c_A \in \Sigma$ for player $A \in \mathsf{Agt}$ if

$$\forall \mathbf{a}_{-A} \in \Sigma^{\mathsf{Agt} \setminus \{A\}} \qquad g_A(b_A, \mathbf{a}_{-A}) < g_A(c_A, \mathbf{a}_{-A})$$

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The only rational issue of the game is (D, D) whose payoff is $(-5, -5)$.

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(Even though this is sub-optimal)

Domination - A finite variant of Cournot competition

	L	M	H
L	(4, 4)	(2, 5)	(1, 3)
M	(5, 2)	(3, 3)	(2, 1)
H	(3, 1)	(1, 2)	(0, 0)

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Action H is strictly dominated by M for Player 1.

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Action H is strictly dominated by M for Player 2.

Domination - A finite variant of Cournot competition

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L	$(4, 4)$	$(2, 5)$	$(1, 3)$
M	$(5, 2)$	$(3, 3)$	$(2, 1)$
H	$(3, 1)$	$(1, 2)$	$(0, 0)$

The action H can be eliminated for both players.

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As both players are rational and assume that their opponent is rational, we Iterate the Elimination of Strictly Dominated Strategies (IESDS).

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The only rational issue of the game is (M, M) whose payoff is $(3,3)$.

Back to Cournot competition

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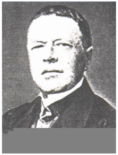
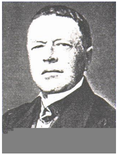


Back to Cournot competition

$$\frac{1}{\beta}\,\mathrm{Profit}_{A_1}(x, y) = -x^2 + x\left(\frac{\alpha - \gamma}{\beta} - y\right)$$



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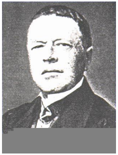

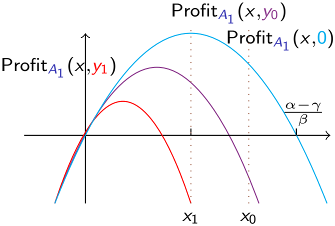

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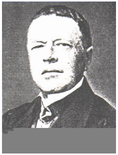

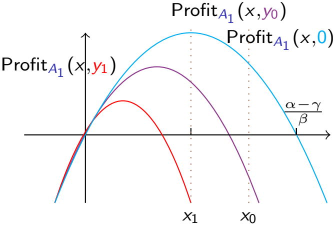

All actions in $\left(\frac{\alpha-\gamma}{2\beta}, \frac{\alpha-\gamma}{\beta}\right]$ are strictly dominated

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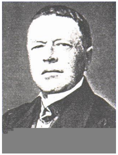

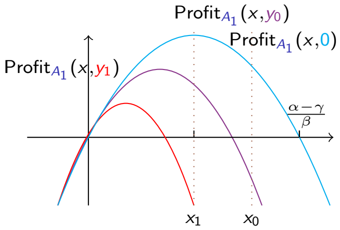

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The IESDS converges to:

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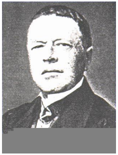

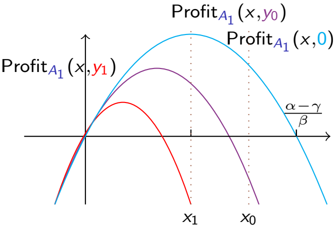

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The result is non trivial: the elimination process is infinite.

Domination - Ice-cream sellers dilemma

The only strategies that are strictly dominated are the two borders...

Summary

We have seen:

- The notion of strictly dominated strategy:
  - + allows to find rational issues of some games,
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  - - not always easy to obtain the rational issue,
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  - - very strong notion: rational issues are not always obtained.
    Ice-cream sellers dilemma

$\rightsquigarrow$ We need another notion to determine rational issues.

Outline

Stability: the concept of Nash equilibria

Nash equilibrium

Let $\left(\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}}\right)$ be a strategic game and $\mathbf{b} \in \Sigma^{\mathsf{Agt}}$ be a strategy profile. We say that $\mathbf{b}$ is a Nash equilibrium iff

$$\forall A \in \mathsf{Agt},\ \forall d_A \in \Sigma \text{ s.t. } g_A(\mathbf{b}_{-A}, d_A) \leq g_A(\mathbf{b})$$

A rational player should not deviate from the Nash equilibrium.

[Nash50] Equilibrium Points in *n*-Person Games (1950).

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	L	R
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B	$(3, 2)$	$(1, 2)$

- R dominates L (but not strictly)
- (B, R) is not a Nash equilibrium, but (T, R) is a Nash equilibrium
- R might not be the best option...
- (B, L) is optimal, hence a Nash equilibrium

[Nash50] Equilibrium Points in *n*-Person Games (1950).

Back to IESDS

	L	M	H
L	(4, 4)	(2, 5)	(1, 3)
M	(5, 2)	(3, 3)	(2, 1)
H	(3, 1)	(1, 2)	(0, 0)

- The only rational issue (M, M) is a Nash equilibrium

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General principle/result

- No strictly dominated action can take part to a Nash equilibrium; this is also the case in the IESDS process
- A profile obtained by IESDS is a Nash equilibrium

Do all the finite matrix games have a Nash equilibrium?

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No!

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No!

The matching penny game

	a	b
a	$(1, 0)$	$(0, 1)$
b	$(0, 1)$	$(1, 0)$

Mixed strategies

Given E, we denote $\Delta(E)$ the set of probability distributions over E.

Mixed strategies

Mixed strategy

If Σ is the of actions (or strategies), $\Delta(\Sigma)$ is the set of mixed strategies.

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Expected payoff

Let $\sigma = (\sigma_{A_1}, \ldots, \sigma_{A_n})$ be a mixed strategy profile. Let $A \in \mathsf{Agt}$:

$$\widetilde{g}_A(\sigma) = \sum_{\mathbf{b}=(b_A)_{A\in\mathsf{Agt}}\in\Sigma^{\mathsf{Agt}}} \underbrace{\left(\prod_{A\in\mathsf{Agt}} \sigma_A(b_A)\right)}_{\text{probability of } \mathbf{b}} g_A(\mathbf{b})$$

is the expected payoff of player A.

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Mixed extension of game G

$\widetilde{G} \stackrel{\text{def}}{=} \left(\mathrm{Agt}, \Delta(\Sigma), (\widetilde{g}_A)_{A\in\mathrm{Agt}}\right)$ is a game.

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Mixed extension of game G

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G has a mixed Nash equilibrium iff $\widetilde{G}$ has a Nash equilibrium.

Nash equilibria in mixed strategies

	a	b
a	$(1, 0)$	$(0, 1)$
b	$(0, 1)$	$(1, 0)$

The following profile is a Nash equilibrium in mixed strategies:

$$\sigma_{A_1} = \frac{1}{2} \cdot \mathtt{a} + \frac{1}{2} \cdot \mathtt{b} \quad \text{and} \quad \sigma_{A_2} = \frac{1}{2} \cdot \mathtt{a} + \frac{1}{2} \cdot \mathtt{b}$$

whose expected payoff is $(\frac{1}{2}, \frac{1}{2})$.

[Nash50] Equilibrium Points in *n*-Person Games (1950).

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Nash Theorem [Nash50]

Any finite game admits mixed Nash equilibria.

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Example: Prisoner dilemma

	C	D
C	$(-3, -3)$	$(-10, 0)$
D	$(0, -10)$	$(-5, -5)$

A best response (for Prisoner 1) to C is

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- Best response correspondence of Player A

$$\begin{aligned} \mathrm{BR}_A : \Sigma^{\mathrm{Agt} \setminus \{A\}} &\to \mathcal{P}(\Sigma) \\ \mathbf{a}_{-A} &\to \{b_A \mid b_A \text{ is a best response to } \mathbf{a}_{-A}\} \end{aligned}$$

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Best response and Nash equilibrium

Proposition

Let $\mathbf{a}$ be a strategy profile.

$$\mathbf{a} \text{ is a Nash equilibrium} \quad \text{if and only if} \quad \mathbf{a} \in \text{BR}(\mathbf{a})$$

An example

	L	R
T	$(1, -1)$	$(0, 0)$
B	$(0, 0)$	$(2, -2)$

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A strategy consists in giving a probability distribution over $\{\mathtt{T}, \mathtt{B}\}$ (resp. $\{\mathtt{L}, \mathtt{R}\}$), that is, it consists in fixing the probability to play T (resp. L).

Assume

$$\sigma_{A_1} = \frac{1}{4} \cdot \mathtt{T} + \frac{3}{4} \cdot \mathtt{B} \qquad \text{and} \qquad \sigma_{A_2} = \frac{1}{2} \cdot \mathtt{L} + \frac{1}{2} \cdot \mathtt{R}$$

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$$g_{A_1}\left(\frac{1}{4}, \frac{1}{2}\right) \;=\; \frac{7}{8} \qquad g_{A_2}\left(\frac{1}{4}, \frac{1}{2}\right) \;=\; -\frac{7}{8}$$

An example

	L	R
T	$(1, -1)$	$(0, 0)$
B	$(0, 0)$	$(2, -2)$

In general, we have

$$\sigma_{A_1} = \alpha \cdot \mathtt{T} + (1 - \alpha) \cdot \mathtt{B} \qquad \text{and} \qquad \sigma_{A_2} = \beta \cdot \mathtt{L} + (1 - \beta) \cdot \mathtt{R}$$

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$$g_{A_1}(\alpha, \beta) \;=\; \alpha(3\beta - 2) - 2\beta + 2 \;=\; -g_{A_2}(\alpha, \beta)$$

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$$g_{A_1}(\alpha, \beta) = \alpha(3\beta - 2) - 2\beta + 2$$

$$\mathrm{BR}_{A_1}(\beta) = \begin{cases} \end{cases}$$

An example

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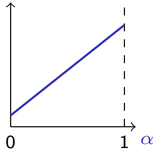
$$g_{A_1}(\alpha, \beta) = \alpha(3\beta - 2) - 2\beta + 2$$

$$\mathrm{BR}_{A_1}(\beta) = \begin{cases} \{1\} & \text{if } 3\beta - 2 > 0 \\ & \\ & \end{cases}$$



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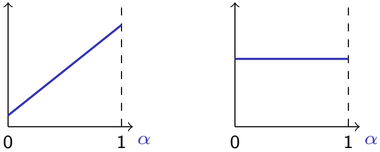

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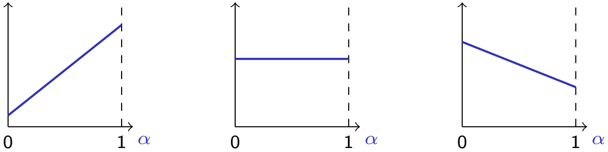
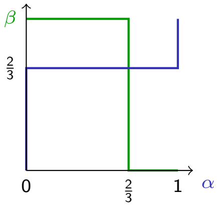


An example

	L	R
T	$(1, -1)$	$(0, 0)$
B	$(0, 0)$	$(2, -2)$

$$\mathrm{BR}_{A_1}(\beta) = \begin{cases} \{1\} & \text{if } 3\beta - 2 > 0 \\ [0,1] & \text{if } 3\beta - 2 = 0 \\ \{0\} & \text{if } 3\beta - 2 < 0 \end{cases} \qquad \mathrm{BR}_{A_2}(\alpha) = \begin{cases} \{1\} & \text{if } 3\alpha - 2 < 0 \\ [0,1] & \text{if } 3\alpha - 2 = 0 \\ \{0\} & \text{if } 3\alpha - 2 > 0 \end{cases}$$



An example

	L	R
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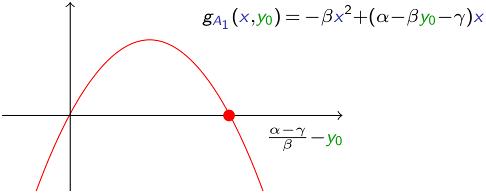
Thus the following profile is an equilibrium in mixed strategies:

$$\sigma_{A_1} = \frac{2}{3} \cdot \texttt{T} + \frac{1}{3} \cdot \texttt{B} \qquad \text{and} \qquad \sigma_{A_2} = \frac{2}{3} \cdot \texttt{L} + \frac{1}{3} \cdot \texttt{R}$$

whose expected payoff is:

$$\left(\frac{2}{3}, -\frac{2}{3}\right)$$

Best response – Back to Cournot competition



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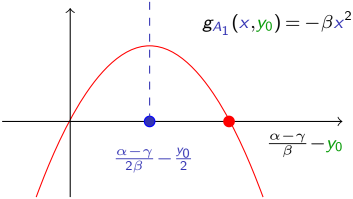

$g_{A_1}(x,y_0) = -\beta x^2 + (\alpha - \beta y_0 - \gamma)x$

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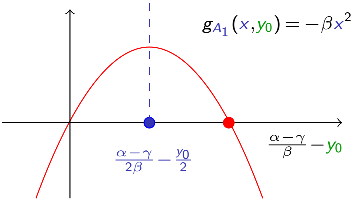

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$$\mathrm{BR}_{A_1}(y_0) = \begin{cases} \left\{ \frac{\alpha-\gamma}{2\beta} - \frac{y_0}{2} \right\} & \text{if } \frac{\alpha-\gamma}{\beta} \geq y_0 \\ \{0\} & \text{otherwise} \end{cases}$$

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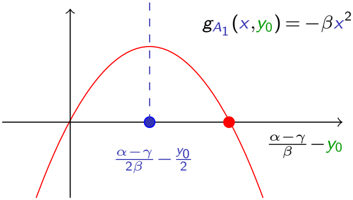

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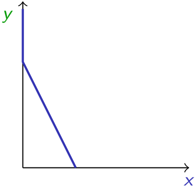
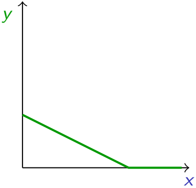
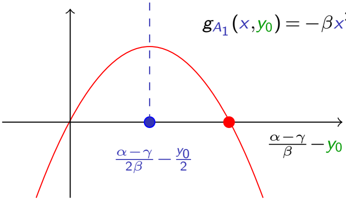


$\mathrm{BR}_{A_1}(y)$



$\mathrm{BR}_{A_2}(x)$

Best response – Back to Cournot competition



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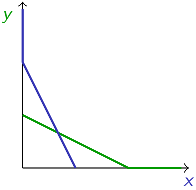

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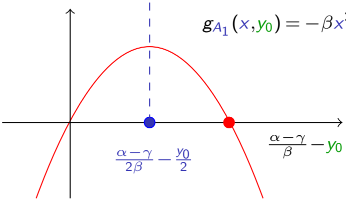

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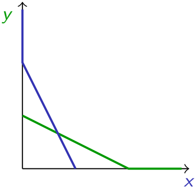

$$(x, y) \in \mathrm{BR}(x, y) \quad \text{iff} \quad x \in \mathrm{BR}_{A_1}(y) \text{ and } y \in \mathrm{BR}_{A_2}(x)$$

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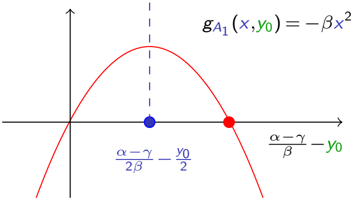

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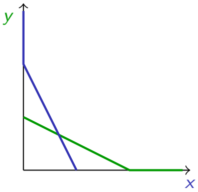

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Best response – Back to the ice-cream sellers dilemma

One can show that the only Nash equilibrium is:

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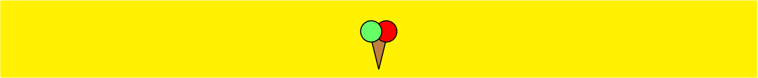

Best response and Nash equilibrium

Proposition

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Nash Theorem [Nash50]

Any finite game admits mixed Nash equilibria.

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Key ingredient of the proof: Brouwer's fixpoint theorem
Or simply Kakutani's fixpoint theorem

Fixpoint theorems

Brouwer's fixpoint theorem

Let $X \subseteq \mathbb{R}^n$ be a convex, compact and nonempty set. Then every continuous function $f \colon X \to X$ has a fixpoint.

Kakutani's fixpoint theorem

Let X be a non-empty, compact and convex subset of $\mathbb{R}^n$. Let $f \colon X \to 2^X$ be a set-valued function on X with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in X$. Then f has a fixpoint.

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$\leadsto$ One can obtain twists or generalizations of Nash Theorem (ex: Nash-Glicksberg Theorem on compact sets of actions)

Outline

A chocolate example

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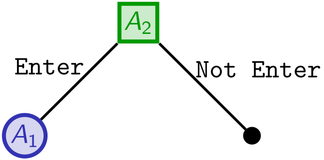

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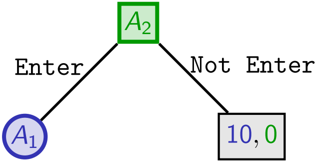

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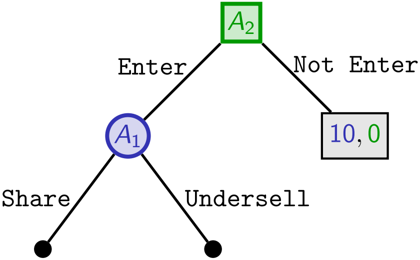
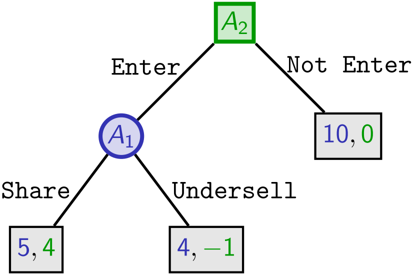
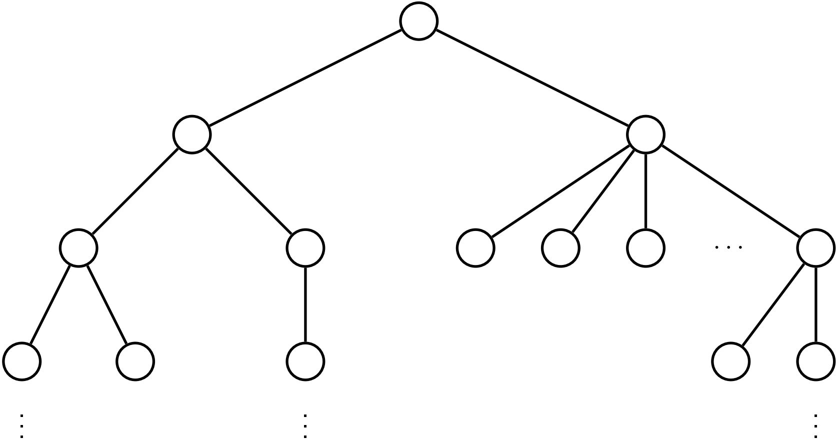


A chocolate example

- A firm A_1 has a monopoly on the production of chocolate.
- Another firm A_2 would like to enter the market of chocolate.
  But entering the market has a cost !
- If A_2 enters the market, then A_1 can share the clients or undersell.



Finite extensive games with perfect information



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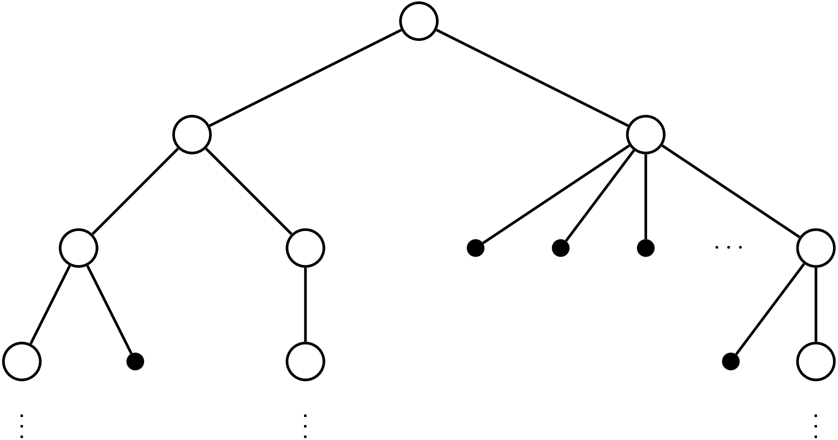

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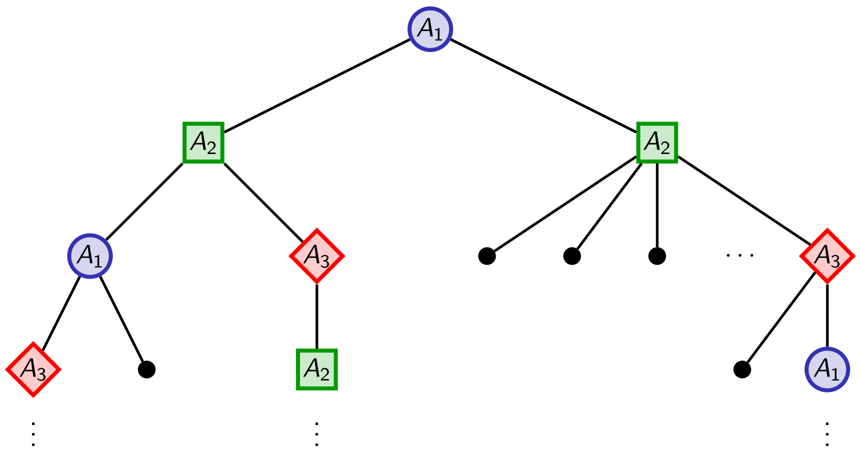

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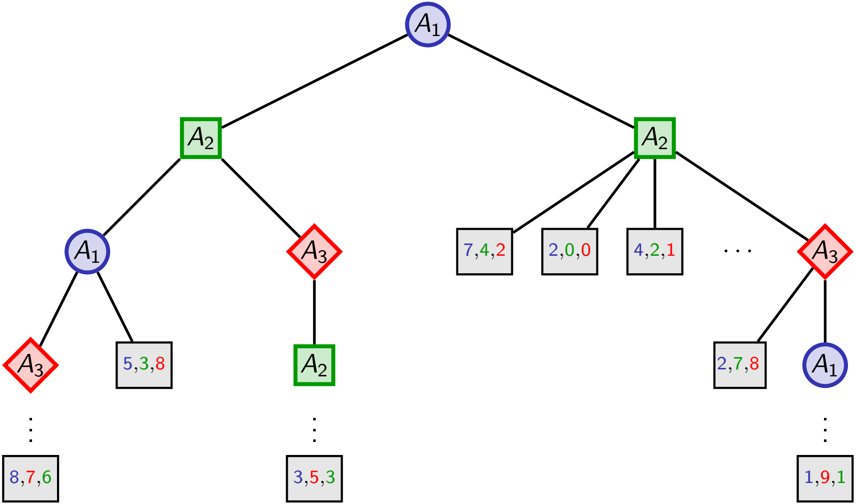
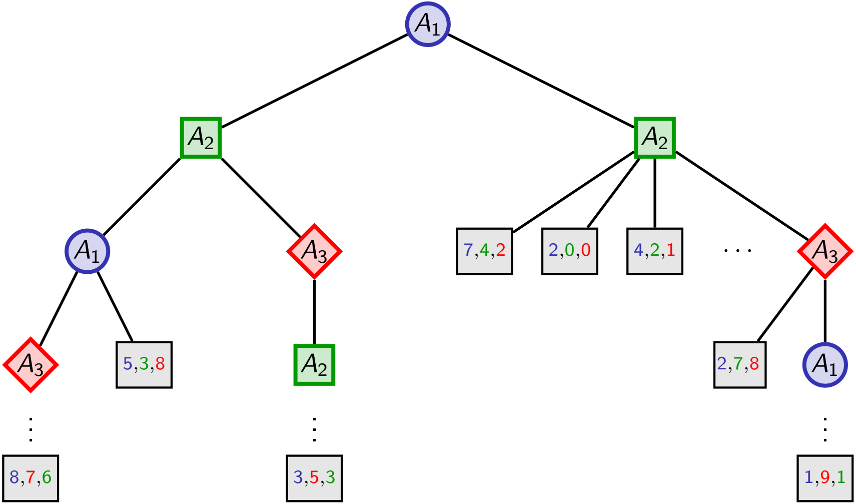


Strategies



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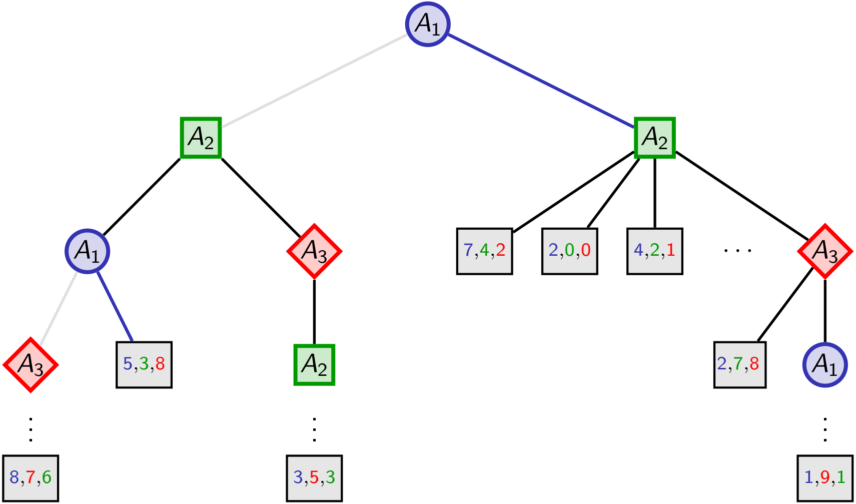
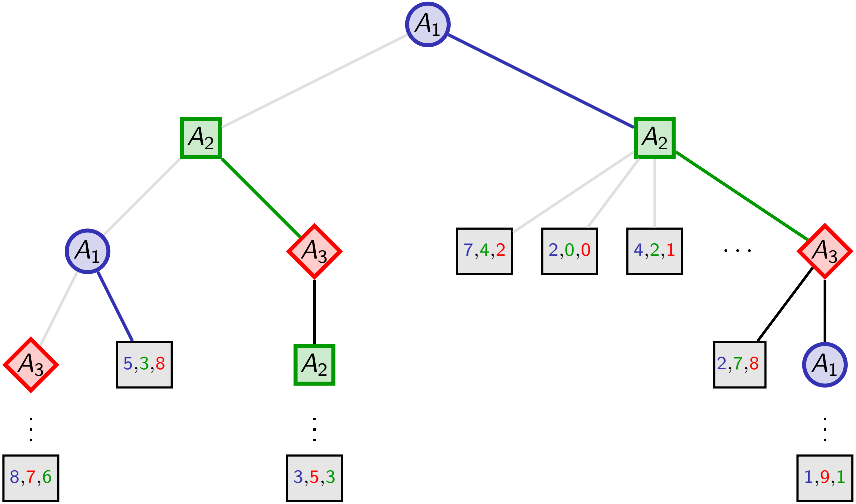


σ_{A_1}: strategy of A_1,

Strategies



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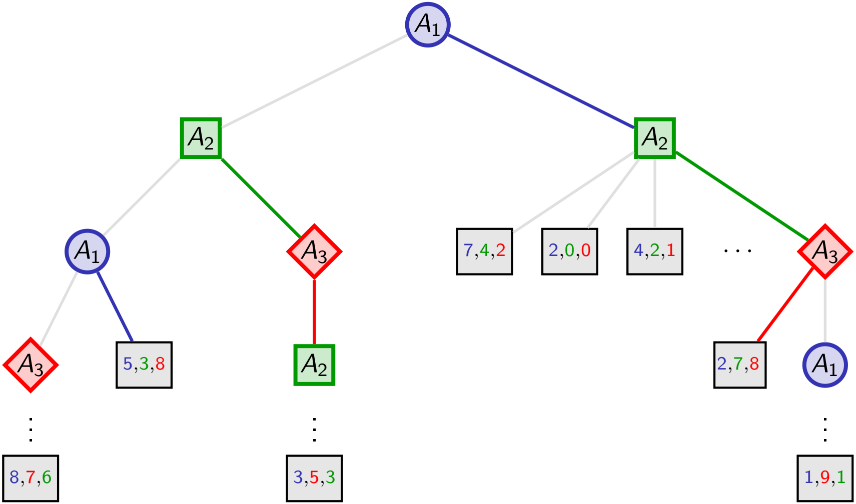
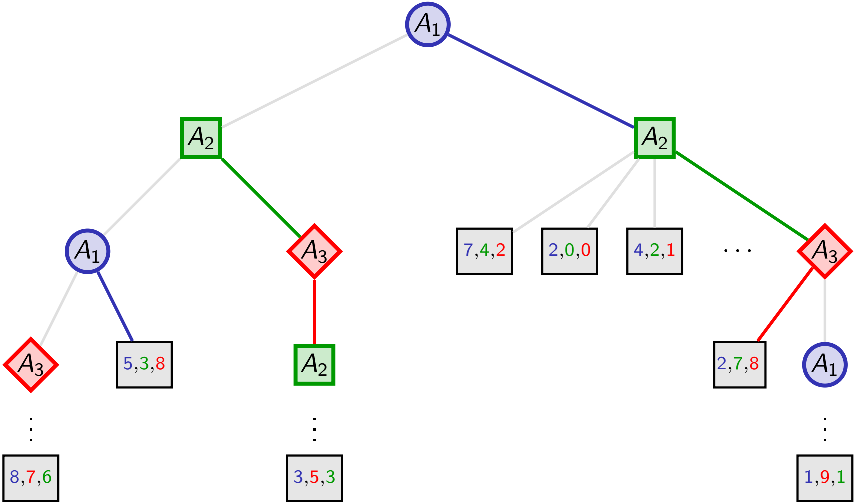


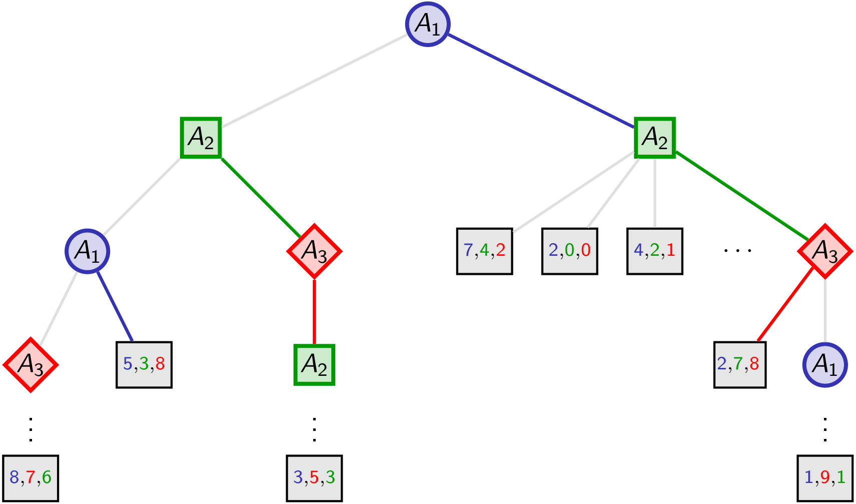
σ_{A_1}: strategy of A_1, σ_{A_2}: strategy of A_2, σ_{A_3}: strategy of A_3.

Strategies



Outcome $(\sigma_{A_1}, \sigma_{A_2}, \sigma_{A_3})$ is the branch determined by the three strategies.

Strategies



One could also have concurrent nodes, or stochastic nodes.
One could also consider randomized strategies.

Extensive games as strategic games

A finite extensive game can always be turned into a strategic game!

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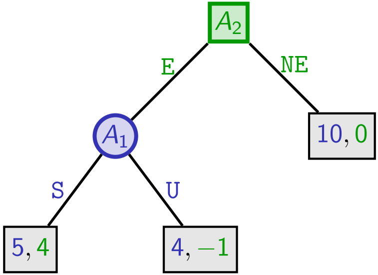

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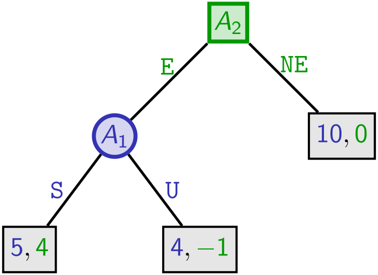

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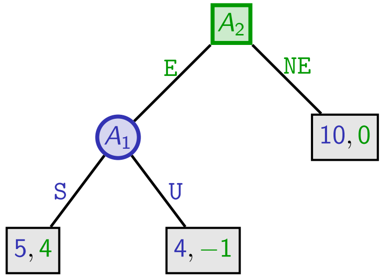

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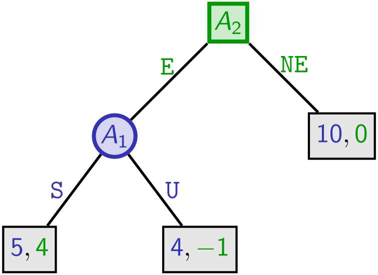

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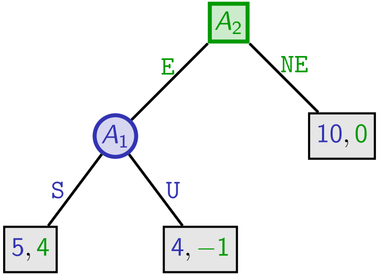

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Under a *perfect recall* hypothesis, mixed and behavior strategies coincide in finite extensive games.

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Corollary

In a finite extensive game (with perfect information), there always exists a Nash equilibrium in behavior strategies.

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Stackelberg competition

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Let a_i denote the quantity produced by the i-th firm.

$$\text{Profit}_{A_1}(a_1, a_2) = a_1 \, \big(\underbrace{\alpha - \beta(a_1 + a_2)}_{\text{selling price}}\big) - \underbrace{\gamma \, a_1}_{\text{production cost}}$$

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What should be the amount of the output to optimize the profit?

Cournot vs Stackelberg (simplified)

VS

$$\text{Profit}_{A_1}(a_1, a_2) = a_1 \; (\alpha - (a_1 + a_2)) - \gamma \, a_1$$

Nash equilibria

- Cournot: $\left(\frac{\alpha-\gamma}{3}, \frac{\alpha-\gamma}{3}\right)$ with payoff $\left(\frac{(\alpha-\gamma)^2}{9}, \frac{(\alpha-\gamma)^2}{9}\right)$.
- Stackelberg: $\left(\frac{\alpha-\gamma}{2}, \frac{\alpha-\gamma}{4}\right)$ with payoff $\left(\frac{(\alpha-\gamma)^2}{8}, \frac{(\alpha-\gamma)^2}{16}\right)$.

Outline

Back to the prisoner dilemma

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C	$(-3, -3)$	$(-10, 0)$
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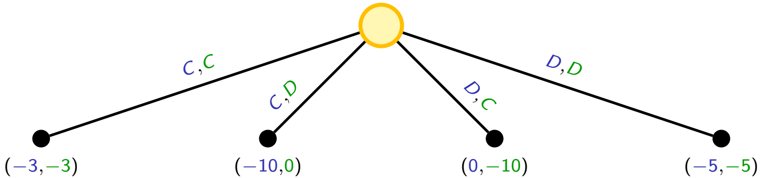
What would happen if the game was repeated again and again?

Prisoner dilemma

As an extensive game with simultaneous moves



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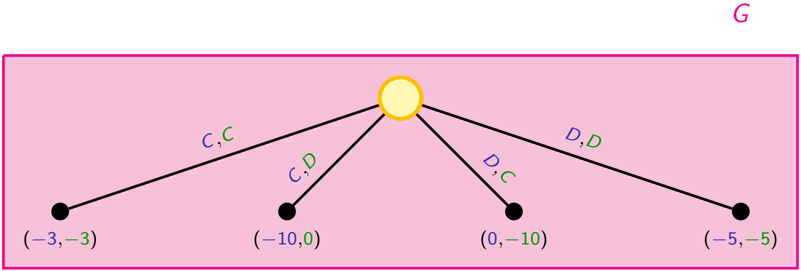
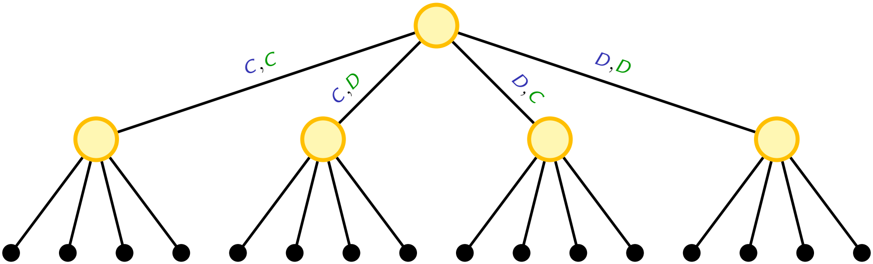


Prisoner dilemma

Repeated twice



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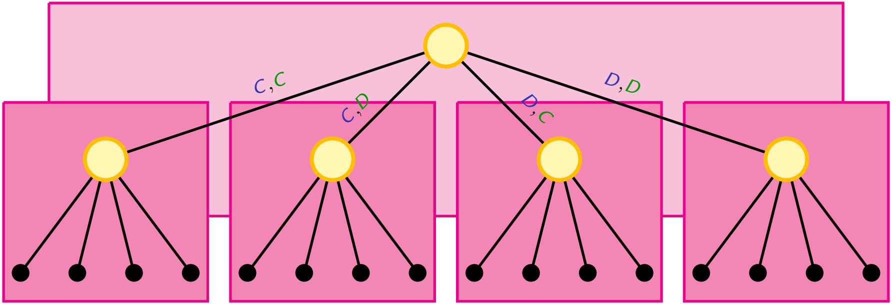
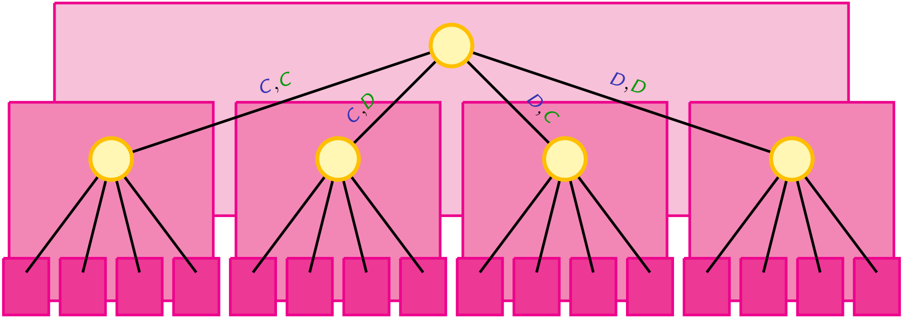


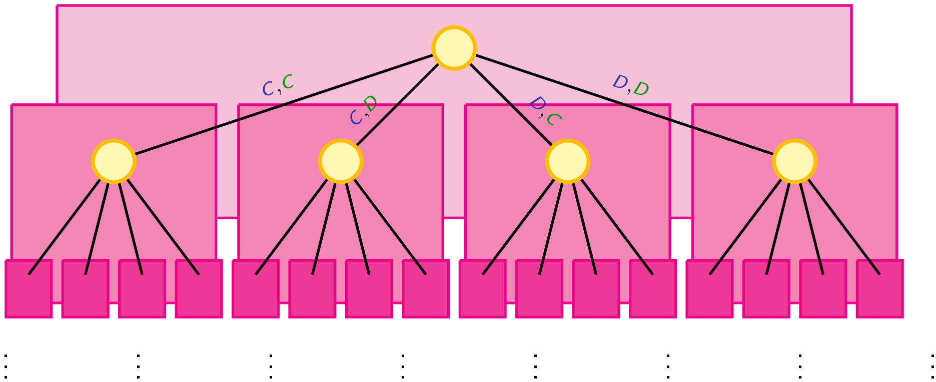
Prisoner dilemma

Repeated three times



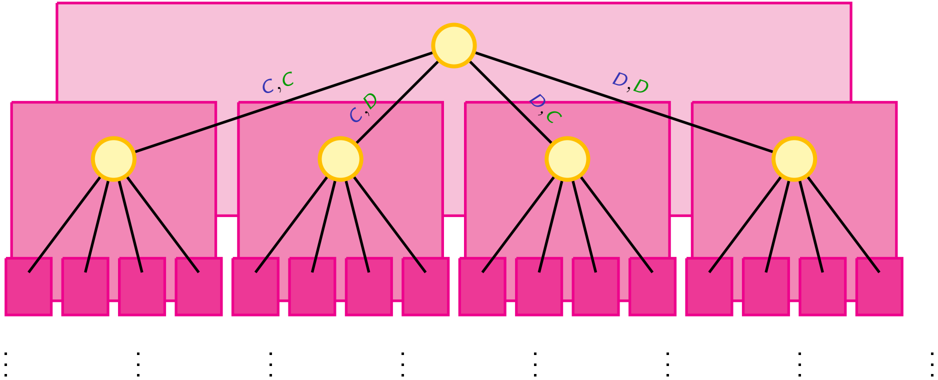
Prisoner dilemma

Repeated infinitely



Prisoner dilemma

Repeated infinitely



We need to define what will be the payoff in such a repeated game

Repeated games

Given $G = (\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}})$ and $t \in \mathbb{N}$,

$\mathbf{a}_t$ denotes the profile of actions played at the t^{th} repetition of G.

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$$g_A^\infty(\mathbf{a}_1, \mathbf{a}_2, \ldots) = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} g_A(\mathbf{a}_t)$$

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Remark

Since repeated games are particular *extensive games with perfect information*, the notion of Nash equilibrium extends.

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We denote by E_T (resp. E_λ and E_∞) the set of payoffs of the Nash equilibria in the game Γ_T (resp. Γ_λ and Γ_∞) in mixed strategies.

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Two approaches to the study of (infinitely) repeated games

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  Study the equilibria of Γ_T and observe what happens when $T \to \infty$
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  Study the equilibria of Γ_λ *and observe what happens when* $\lambda \to 1$
- the uniform approach:
  Study "directly" the equilibria of Γ_∞

Equilibria in repeated games - Prisoner dilemma

	C	D
C	$(-3, -3)$	$(-10, 0)$
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One can prove that:

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- For $\frac{3}{5} \leq \lambda < 1$, $(-3,-3) \in E_\lambda$

- Grim-Trigger strategy: play C as long as everyone plays C; play D otherwise
- Payoff of main outcome: $(-3,-3)$
- Payoff of any deviation $(C,C)\cdots(C,C)(D,C)(-,D)(-,D)\cdots$ is < -3

$\rightsquigarrow$ No profitable deviation

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  - Total-payoff of main outcome: $(3, 3)$
  - No profitable deviation at the second round, since D is dominating
  - What if a player plays D instead of C at the first round? Then, at the second round, he will be punished by P. He would then get at most $3 - 1 = 2$. Not profitable.

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- Not so easy to compute the sets E_T...

The uniform approach I

Minmax level of Player A

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The Minmax level of Player A denoted v_A is defined by:

$$v_A = \min_{\pi_{-A} \in (\Delta(\Sigma)^{\mathrm{Agt} \setminus \{A\}})} \max_{b_A \in \Delta(\Sigma)} g_A(b_A, \pi_{-A}).$$

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C	$(-3,-3)$	$(-10,0)$
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$$v_{A_1} = \min_{\beta} \max_{\alpha} g_{A_1}(\alpha,\beta) = -5 = v_{A_2}$$

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The uniform approach II

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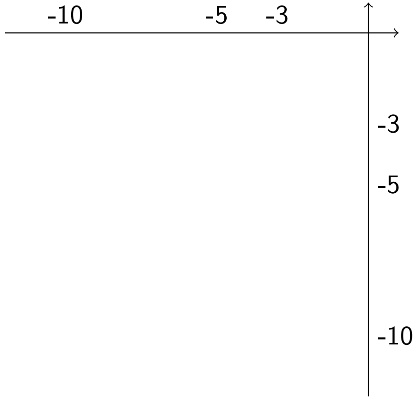

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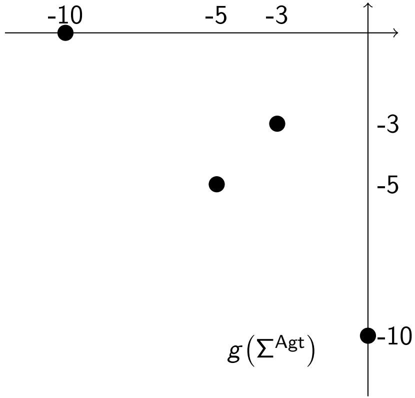

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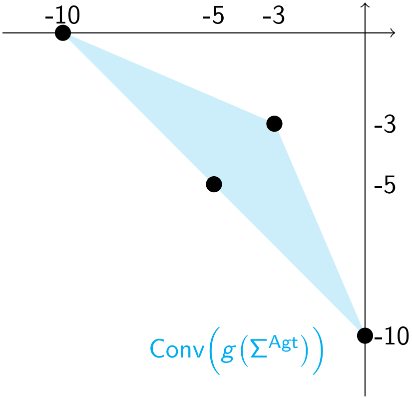

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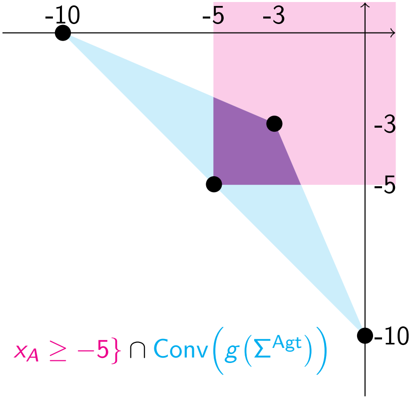

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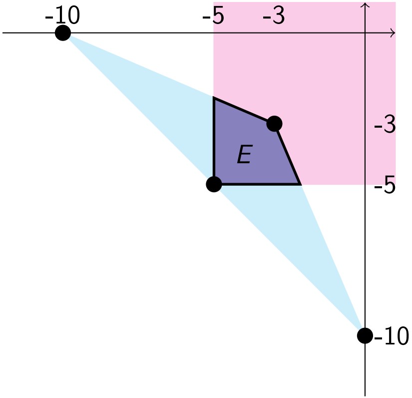

Results on the uniform approach

Folk Theorem [AS76,Rub77]

[AS76] Aumann,Shapley. Long-term competition – A game theoretic analysis *(Essays on Game Theory, 1994)*
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The proofs build “simple” equilibria based on the concept of **punishment**.

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Proof idea of the Folk Theorem

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  - if player A is the first player deviating from this plan, then all players of $\mathsf{Agt} \setminus \{A\}$ switch to π_{-A}

Example: the prisoner dilemma

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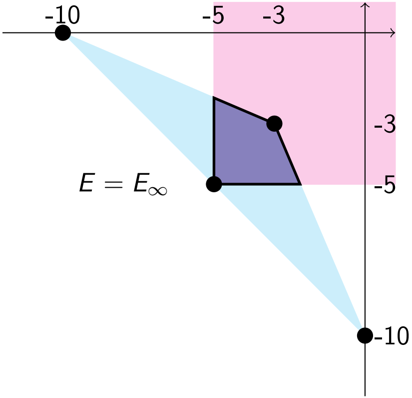

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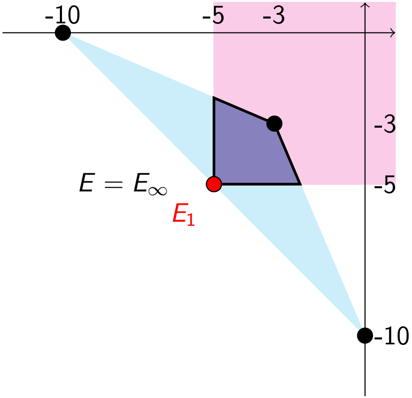
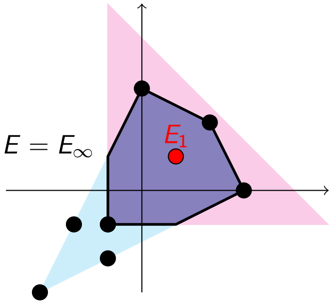


Example: the variant of the prisoner dilemma

	C	D	P
C	$(2, 2)$	$(0, 3)$	$(-2, -1)$
D	$(3, 0)$	$(1, 1)$	$(-1, -1)$
P	$(-1, -2)$	$(-1, -1)$	$(-3, -3)$

We have that $v_{A_1} = v_{A_2} = -1$ and $E_1 = \{(1, 1)\}$.



The compact approach

Link between E_T and E_∞ [BK87]

Given $G = (\mathsf{Agt}, \Sigma, (g_A)_{A \in \mathsf{Agt}})$ satisfying *some condition (easy to test)*,[a] we have:

$$E_T \xrightarrow{T \to \infty} E_\infty$$

[a] For every $A \in \mathsf{Agt}$, there is $\mathbf{b} \in E_1$ s.t. $g_A^T(\mathbf{b}) > v_A$.

Note: The prisoner dilemma does not satisfy the above condition

[Tom06] Tomala. Théorie des jeux : Introduction à la théorie des jeux répétés, chapter "Jeux répétés"
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Link between E_λ and E_∞ [Sor86]

Given $G = (\mathrm{Agt}, \Sigma, (g_A)_{A\in\mathrm{Agt}})$ satisfying *some condition (easy to test)*,[a] we have:

$$E_\lambda \xrightarrow{\lambda\to 1} E_\infty$$

[a] Two players, or there is $\mathbf{x} \in E_\infty$ s.t. $x_A > v_A$ for every $A \in \mathrm{Agt}$.

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Outline

Conclusion

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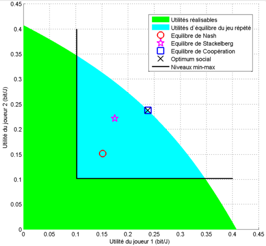
  - includes notions and mechanisms that will be used in models for verification
  - has already interesting applications to the modelling of wireless communications in general, and more specifically to distributed power control problems [LL10]

Definition 1 (Static PC game): The static PC game is a triplet $\mathcal{G} = (\mathcal{K}, \{\mathcal{A}_i\}_{i\in\mathcal{K}}, \{u_i\}_{i\in\mathcal{K}})$ *where* $\mathcal{K}$ *is the set of players,* $\mathcal{A}_1, ..., \mathcal{A}_K$ *are the corresponding sets of actions,* $\mathcal{A}_i = [0, P_i^{\max}]$, $P_i^{\max}$ *is the maximum transmit power for player* i, *and* $u_1, ..., u_k$ *are the utilities of the different players which are defined by:*

$$u_i(p_1, ..., p_K) = \frac{R_i f(\text{SINR}_i)}{p_i} \text{ [bit/J]}. \qquad (3)$$



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Talk on Thursday!

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- Why game theory for verification?
- Which games? How can we treat them?
- Discussion