On finite-memory determinacy of games on graphs

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Based on joint work with Stéphane Le Roux, Youssouf Oualhadj, Mickael Randour, Pierre Vandenhove
(Published at CONCUR'20)

The talk in one slide

Strategy synthesis for two-player games

- Find good and simple controllers for systems interacting with an antagonistic environment

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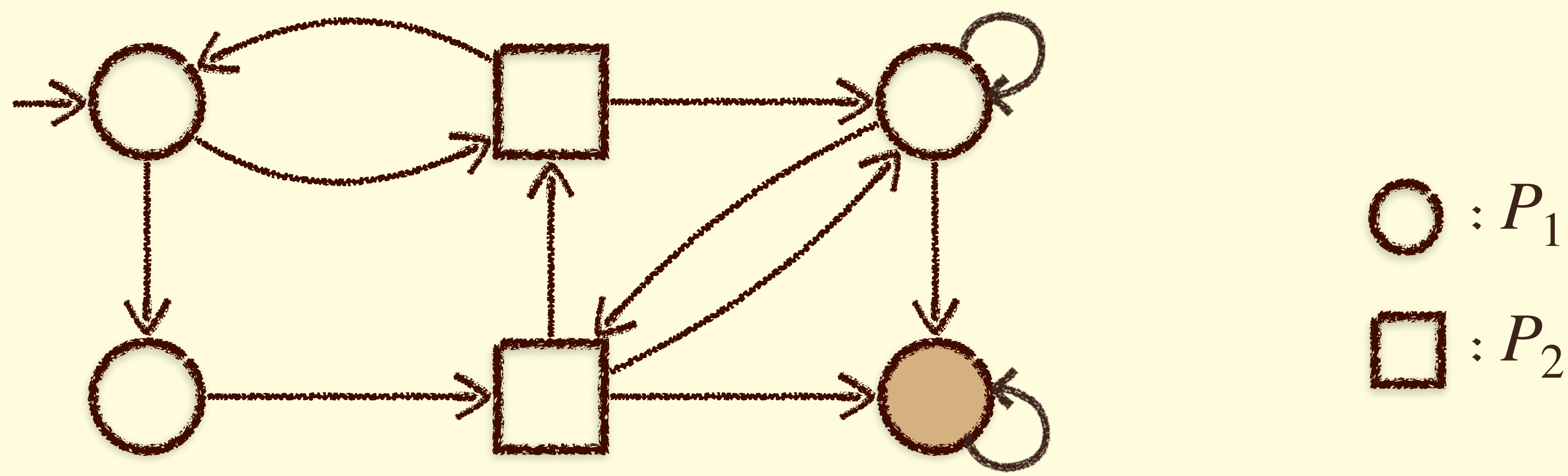
When are simple strategies sufficient to play optimally?

The setting - Example of a game



Reachability winning condition for P_1

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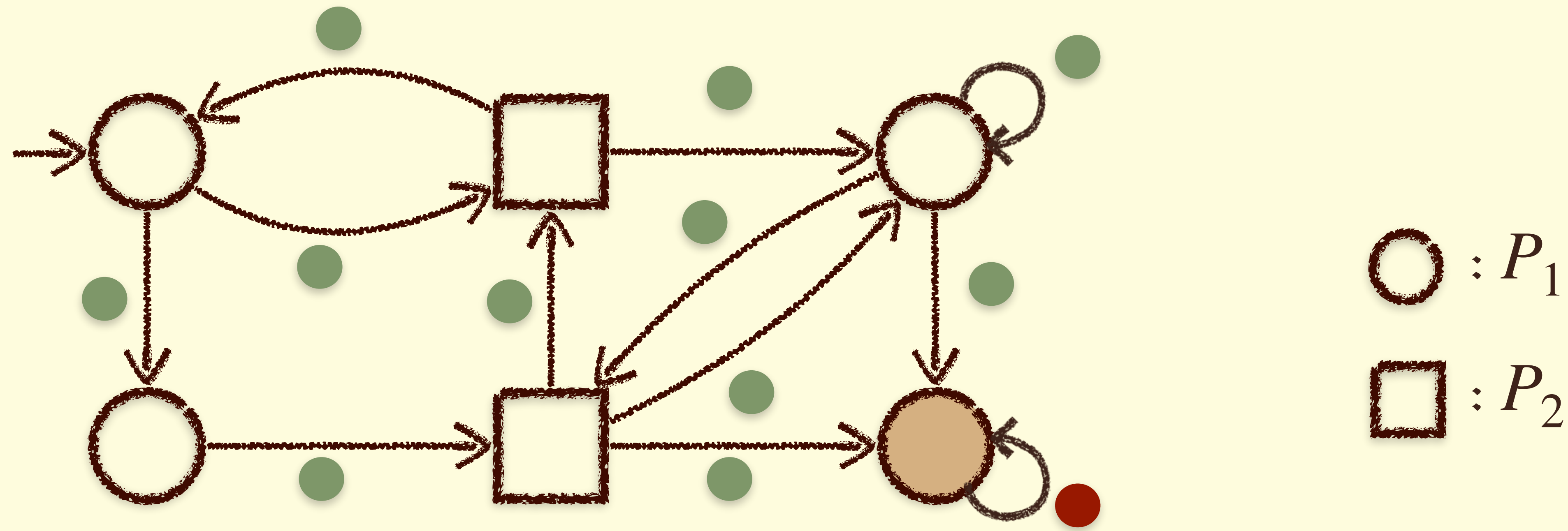
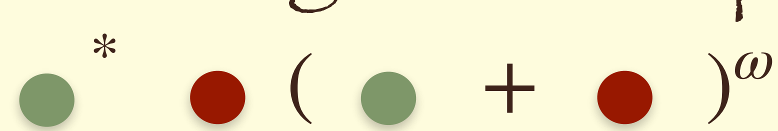


Reachability winning condition for P_1

Use of colors to define winning condition/preference relation



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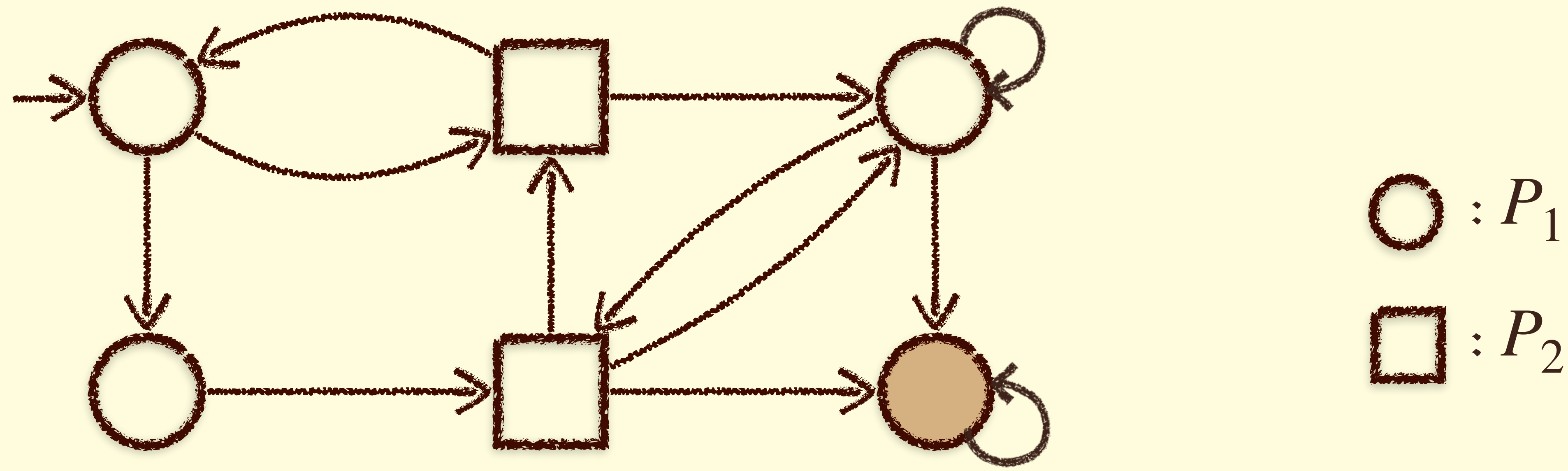

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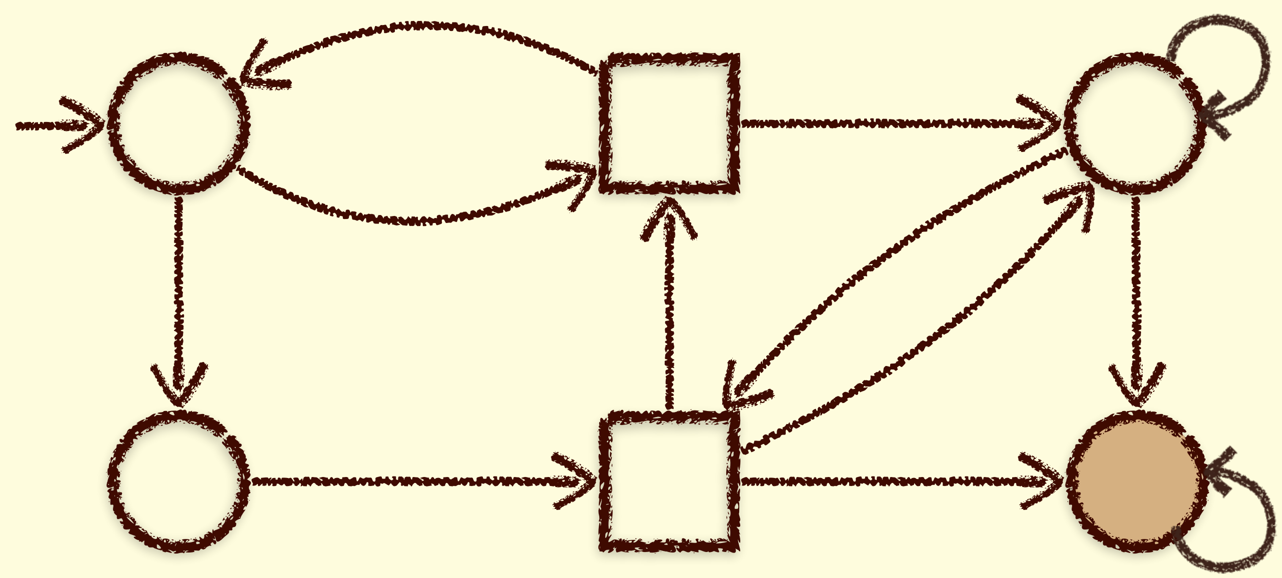

$\bigcirc$: P_1

$\square$: P_2

Reachability winning condition for P_1

The game is played using strategies:

$$\sigma_i : S^*S_i \rightarrow E$$

Families of strategies

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Subclasses of interest

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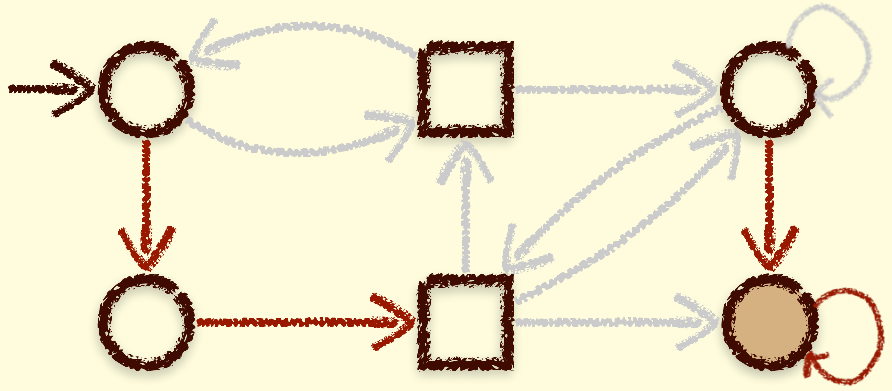
Subclasses of interest

- Memoryless strategy: $\sigma_i : S_i \rightarrow E$



« Reach the target »

Families of strategies

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- Memoryless strategy: $\sigma_i : S_i \to E$
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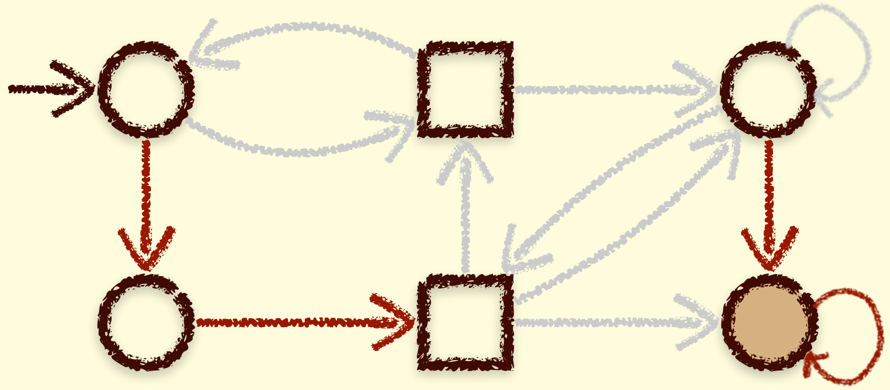

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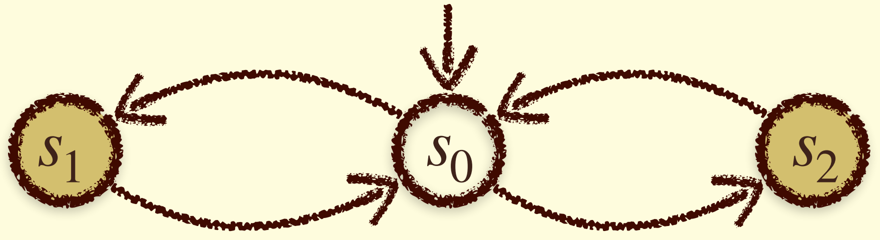

« Visit both s_1 and s_2 »

Every odd visit to s_0, go to s_1

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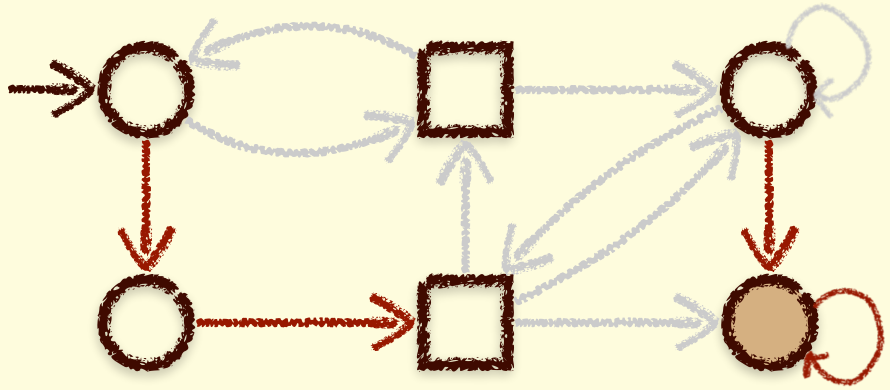

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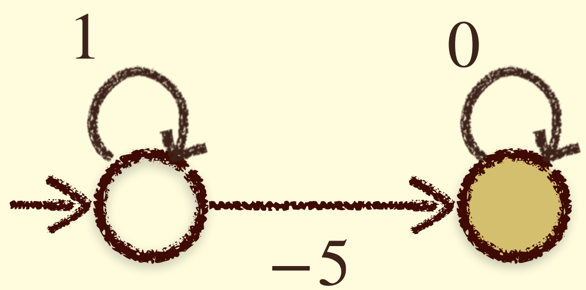
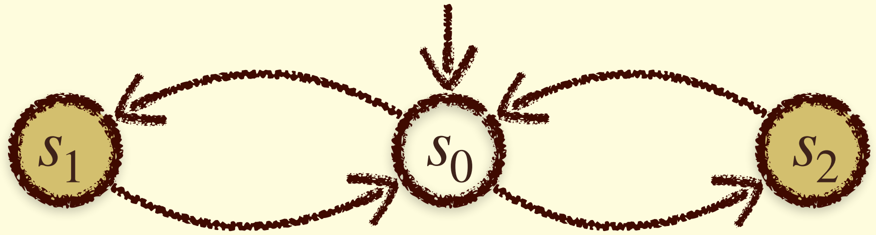


« Reach the target with energy 0 »

Loop 5 times in the initial state



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The setting - Preference relation

A preference relation $\sqsubseteq$ is a total preorder on C^ω.

$\pi \sqsubseteq \pi'$ and $\pi' \sqsubseteq \pi$ means that π and π' are equally appreciated

$\pi \sqsubseteq \pi'$ and $\pi' \not\sqsubseteq \pi$ means that π' is preferred over π

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Examples

- $W \subseteq C^\omega$ winning condition:
  $\pi \sqsubseteq \pi'$ if either $\pi' \in W$ or $\pi \notin W$
- Quantitative real payoff f
  $\pi \sqsubseteq \pi'$ if $f(\pi) \leq f(\pi')$
  Ex: MP, AE, TP

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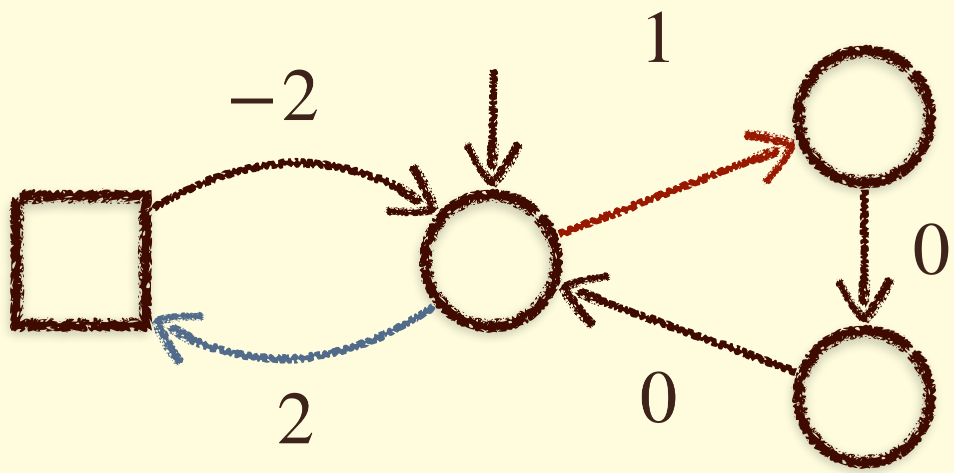
Zero-sum assumption:

- Preference of P_1 is $\sqsubseteq$
- Preference of P_2 is $\sqsubseteq^{-1}$

Payoffs based on energy



Focus on two memoryless strategies

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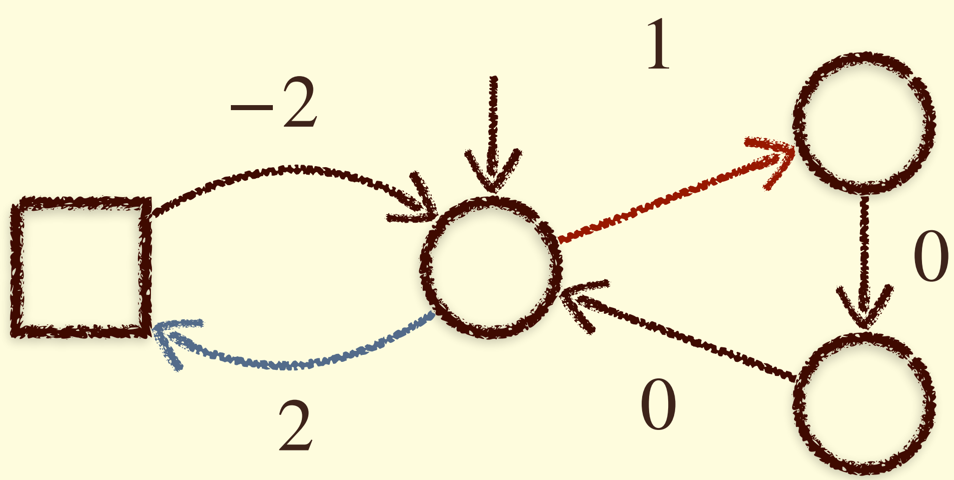

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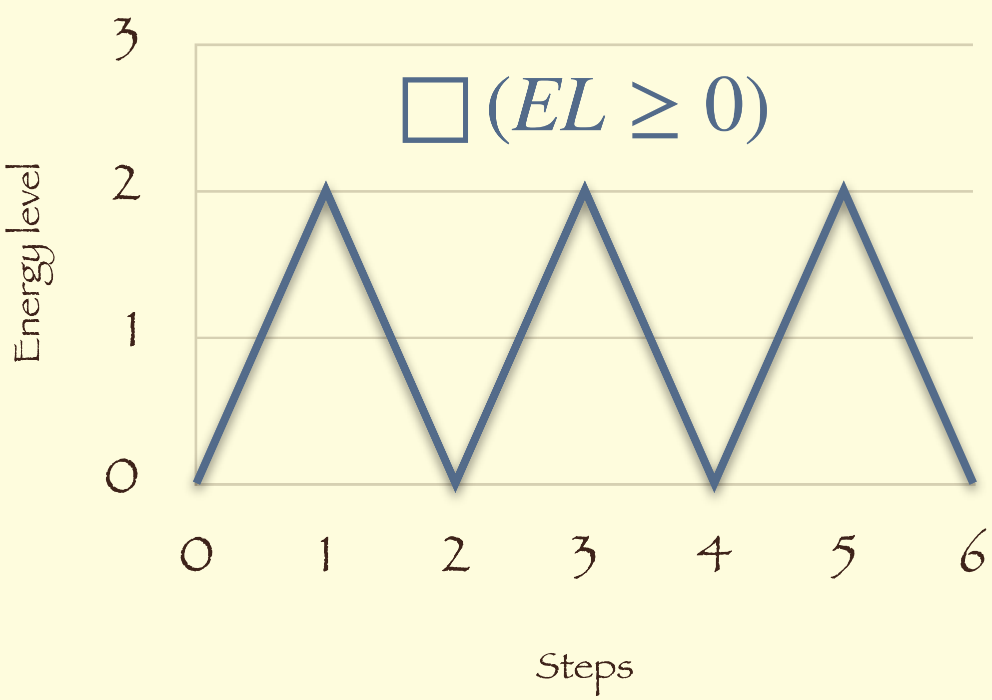
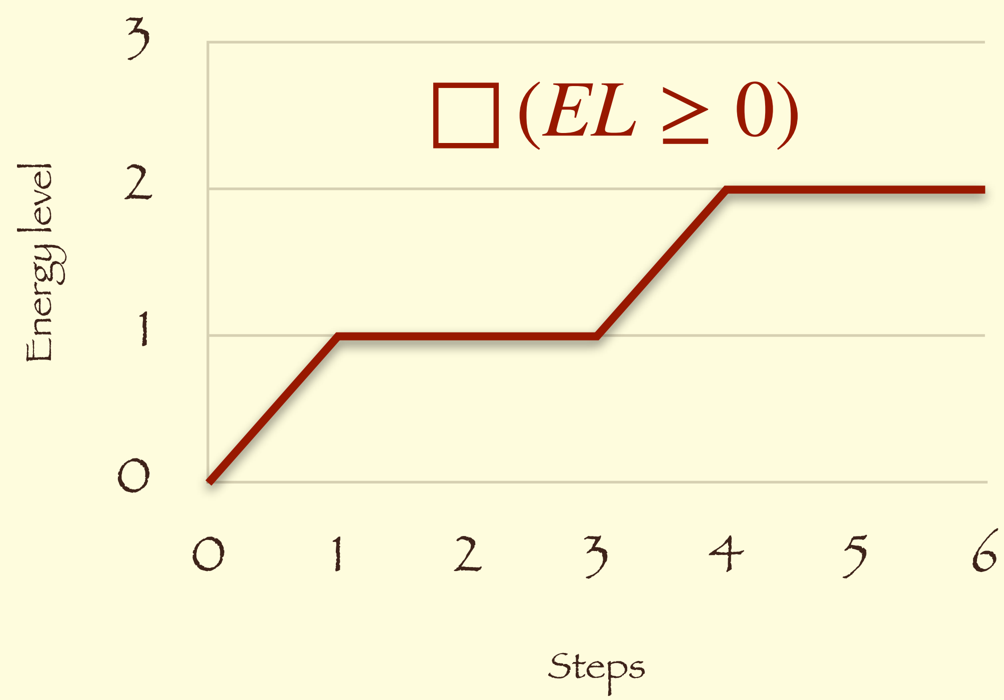
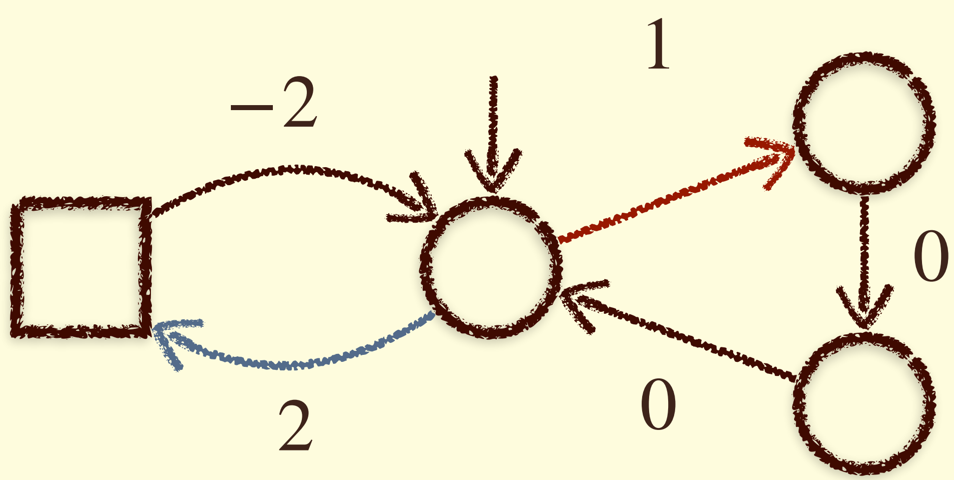




- Constraint on the energy level (EL)

Payoffs based on energy



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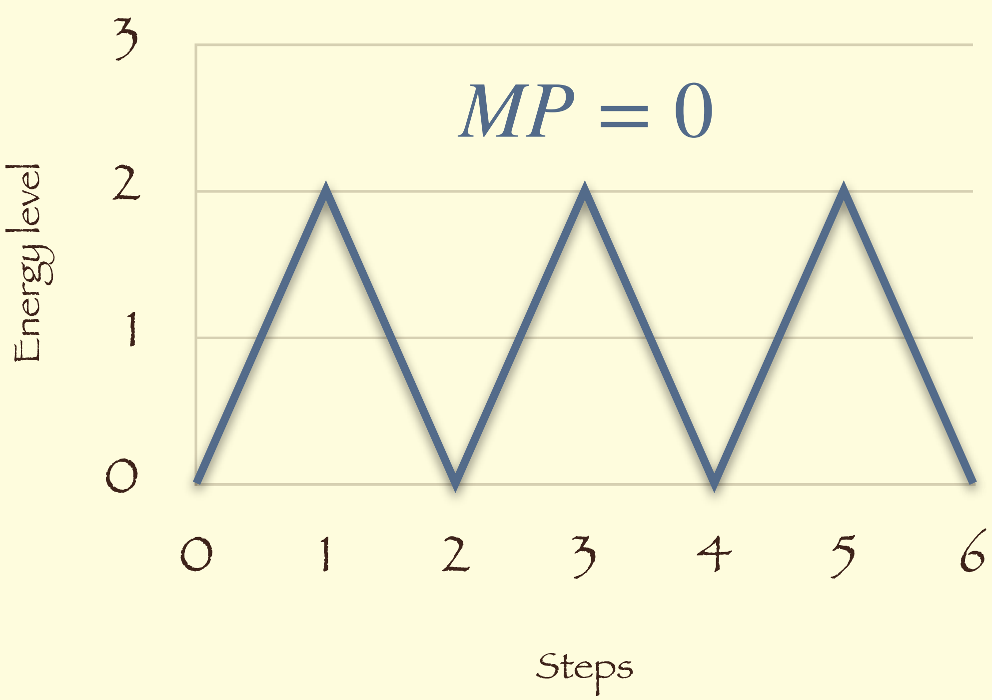

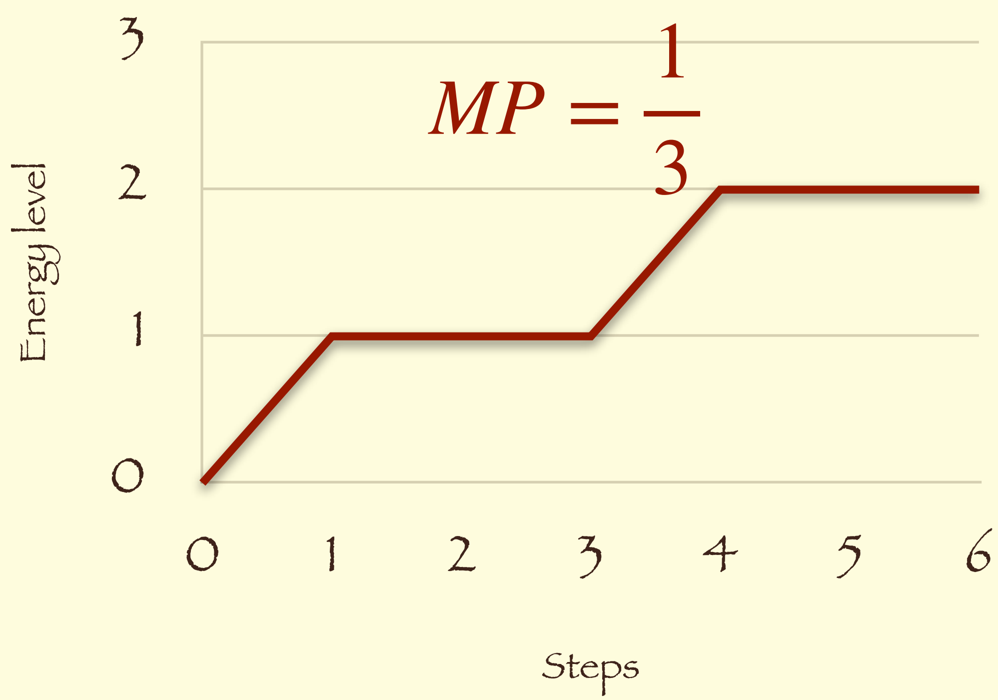

- Constraint on the energy level (EL)
- Mean-payoff (MP): long-run average payoff per transition

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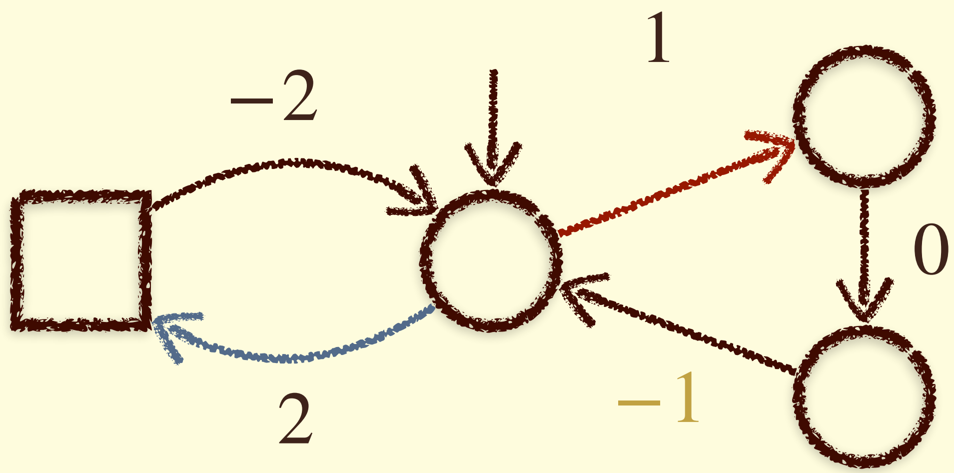

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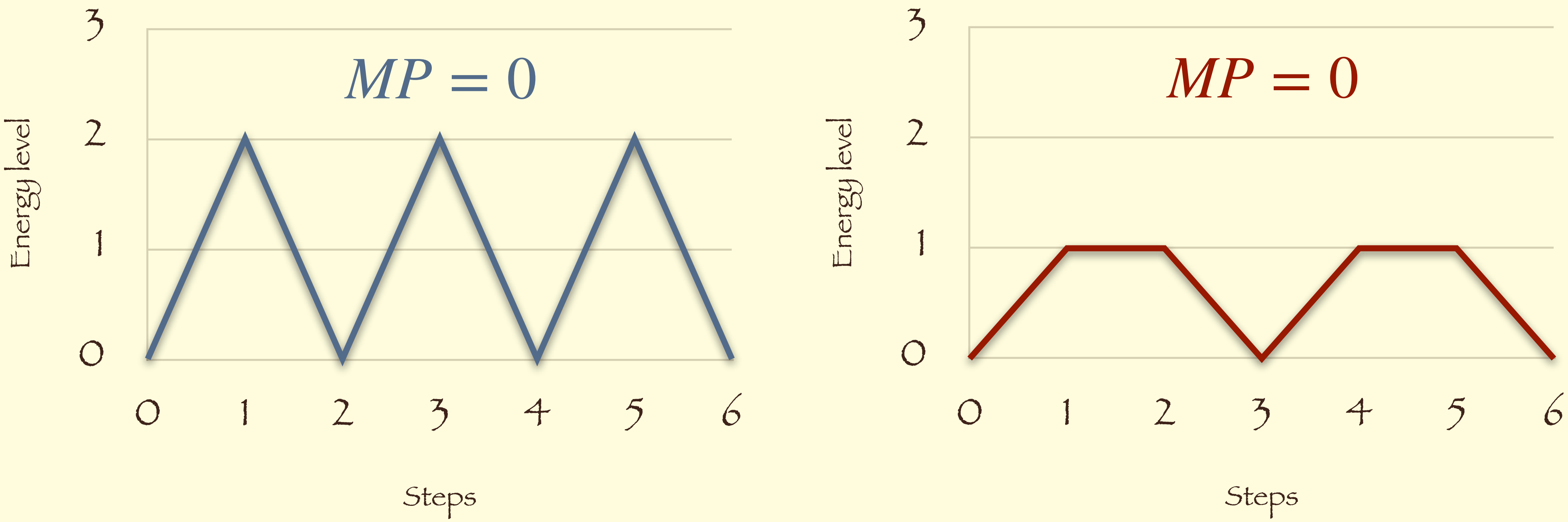

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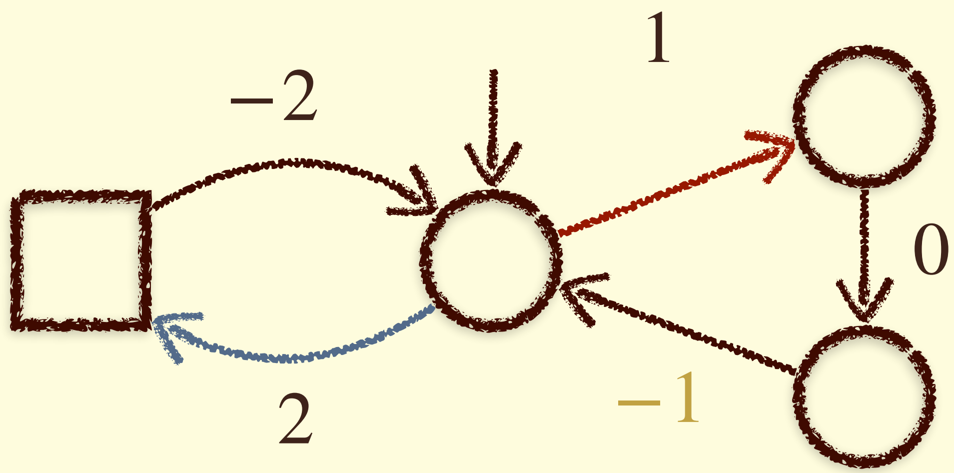

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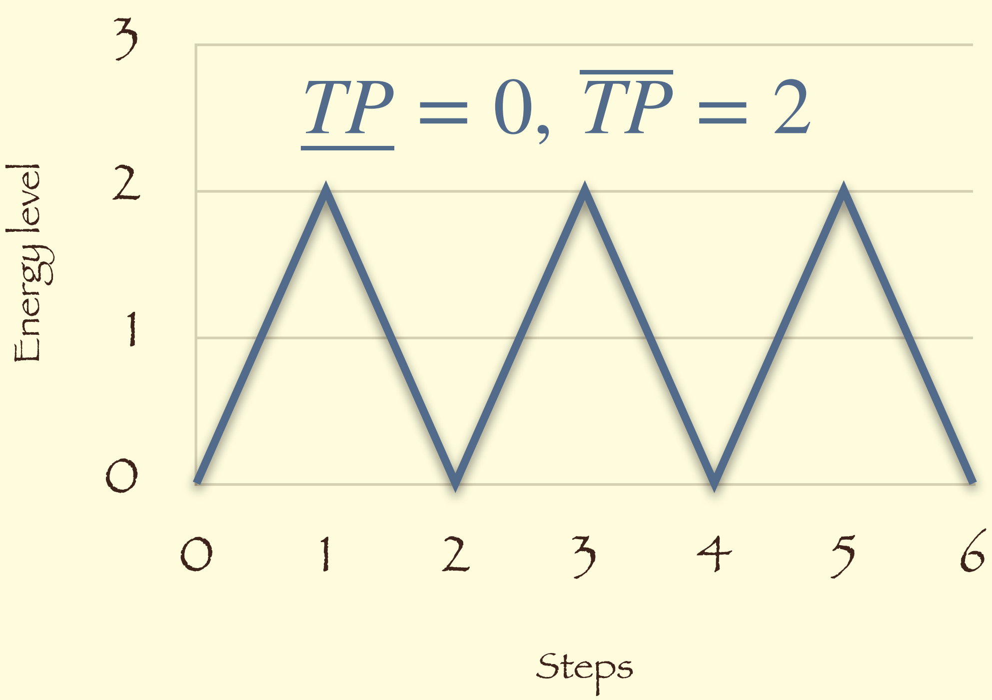

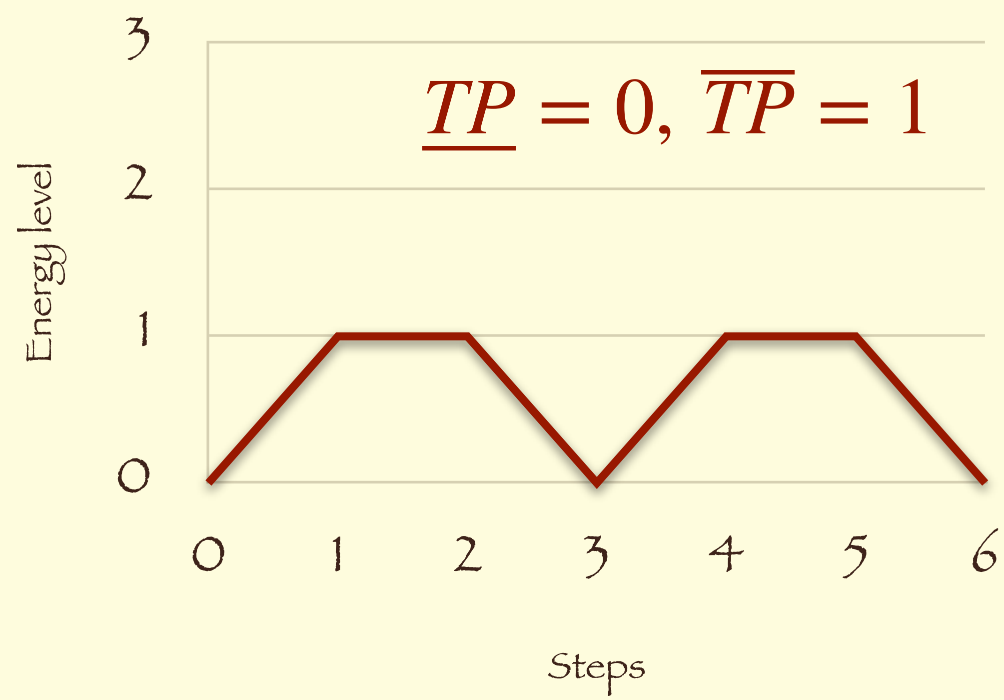

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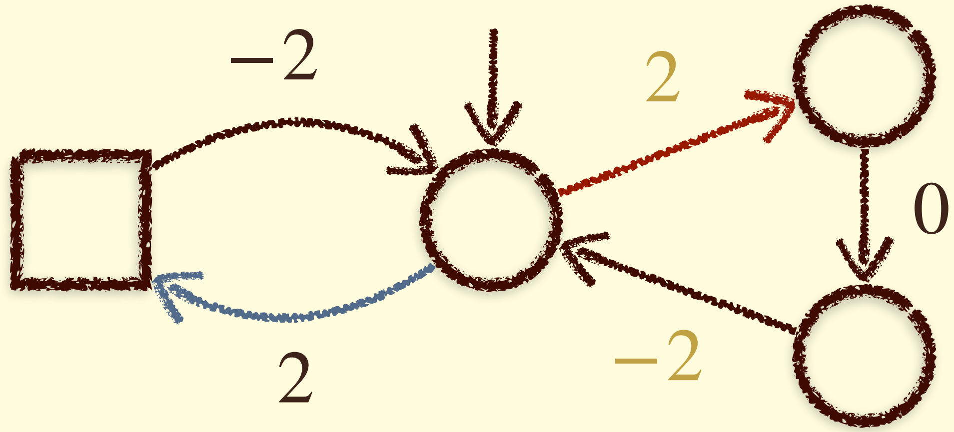
- Mean-payoff (MP): long-run average payoff per transition
- Total-payoff (TP)

Payoffs based on energy



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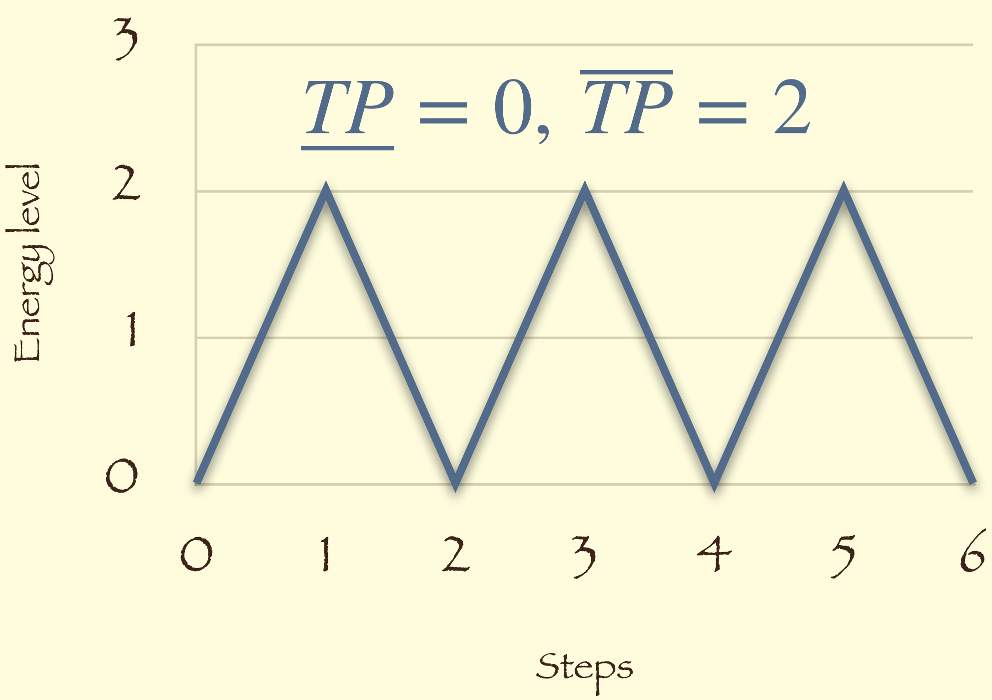

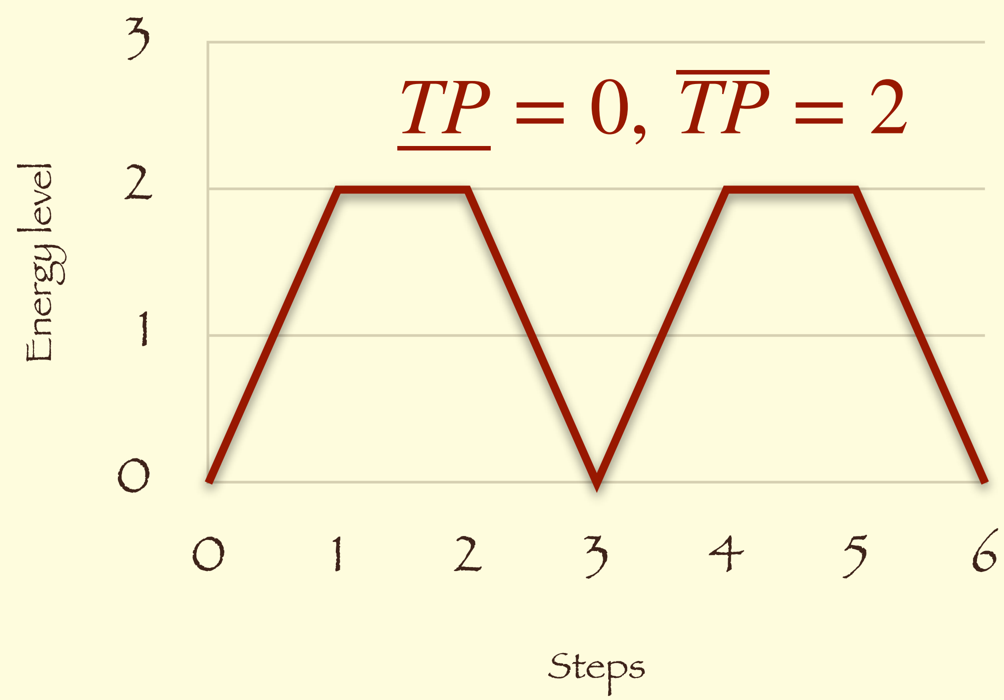

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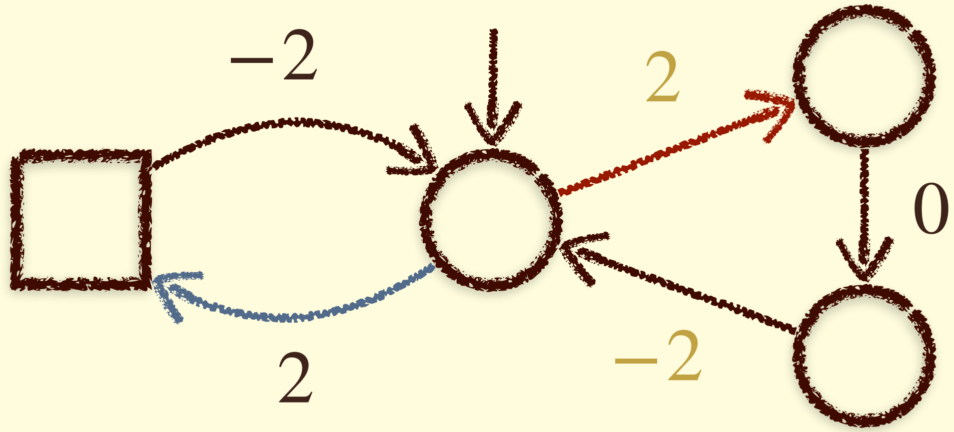

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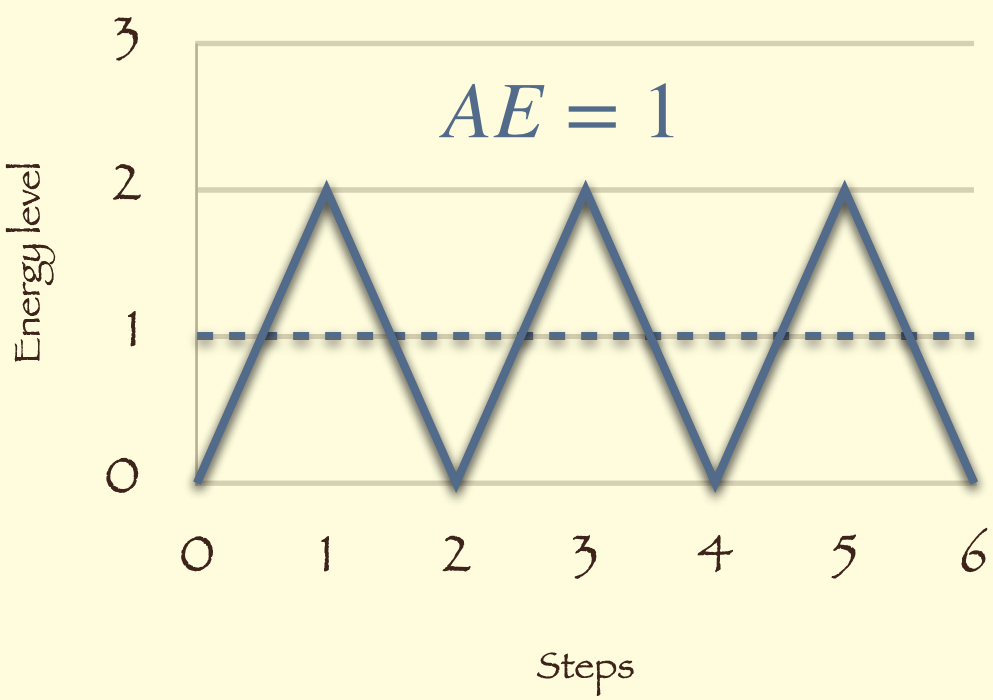

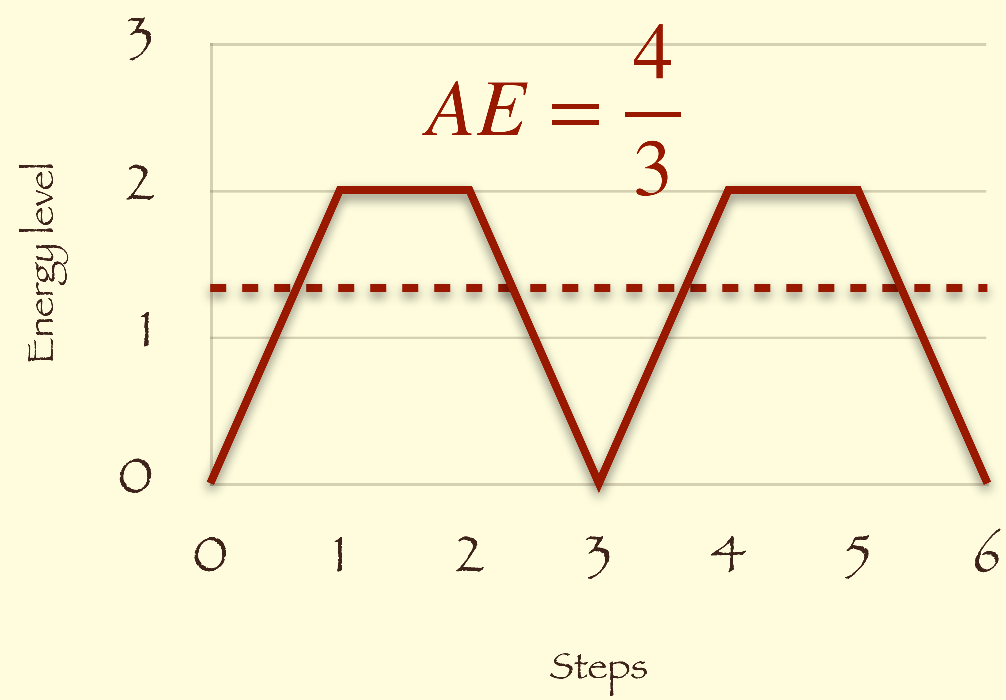

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- Total-payoff (TP)
- Average-energy (AE)

Optimality of strategies

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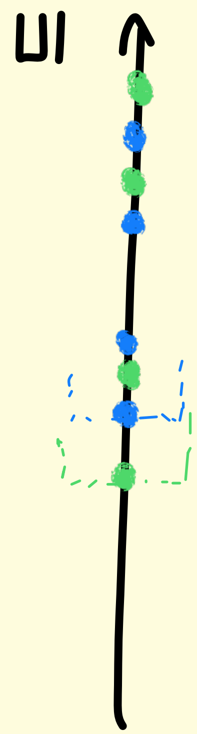

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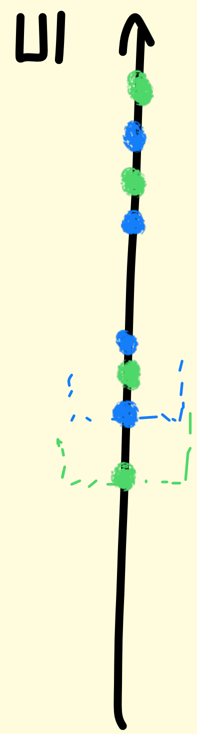

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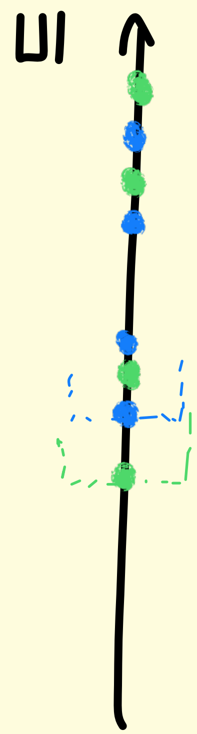

$Out(\sigma_1)^\uparrow \subseteq Out(\sigma_1')^\uparrow$

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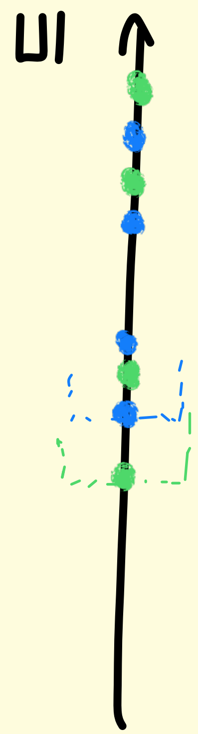
σ_1 optimal whenever it is better than any other σ_1'

Optimality of strategies



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$\Longrightarrow$ σ_1 is better than σ'_1

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Remark

- To be distinguished from:
  - ϵ-optimal
  - Subgame-perfect optimal (in our case: Nash equilibria)

A focus on memoryless strategies

When are memoryless strategies sufficient to play optimally?

Quite often!

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Examples

- Reachability, safety, Büchi, parity, MP, EL ≥ 0, TP, AE, etc...

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Can we characterize when they are?

YES !

And this is a beautiful result by Gimbert and Zielonka, CONCUR'05

The memoryless story

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- Characterization of the preference relations admitting optimal memoryless strategies for both players in all finite games [GZ05]

The Gimbert-Zielonka characterization for memory less determinacy (1)

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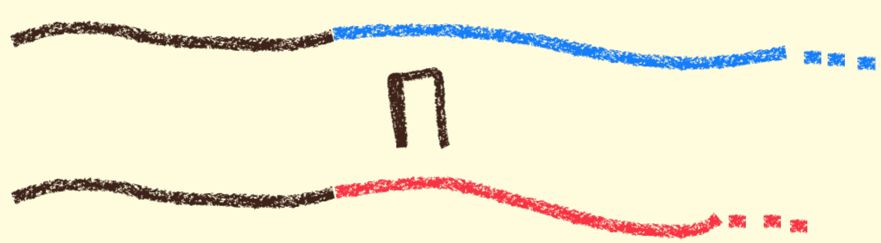

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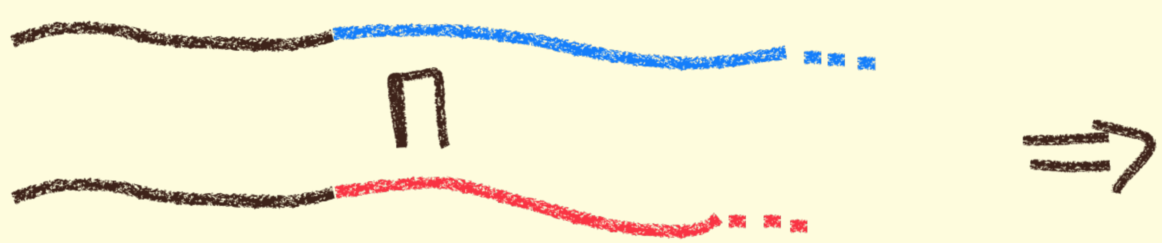

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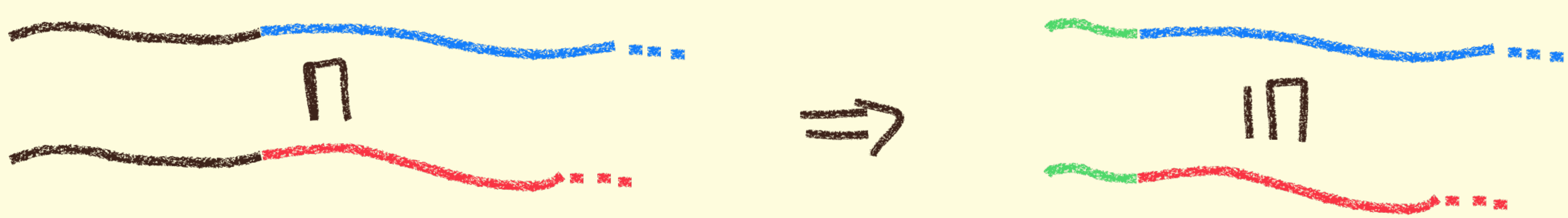

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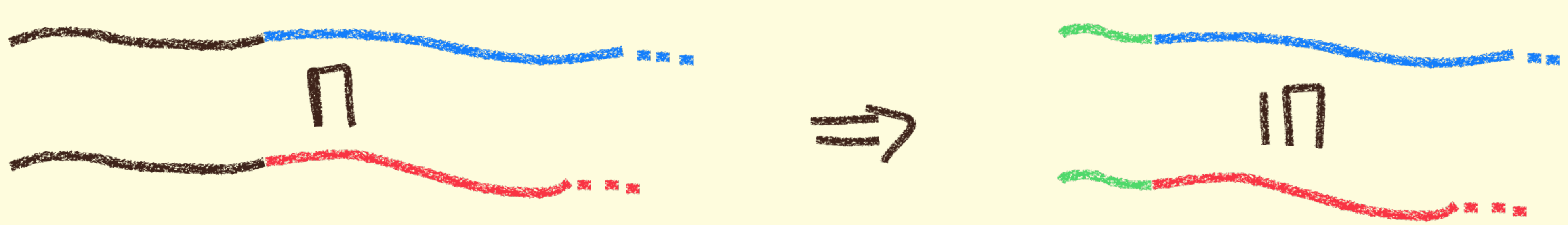

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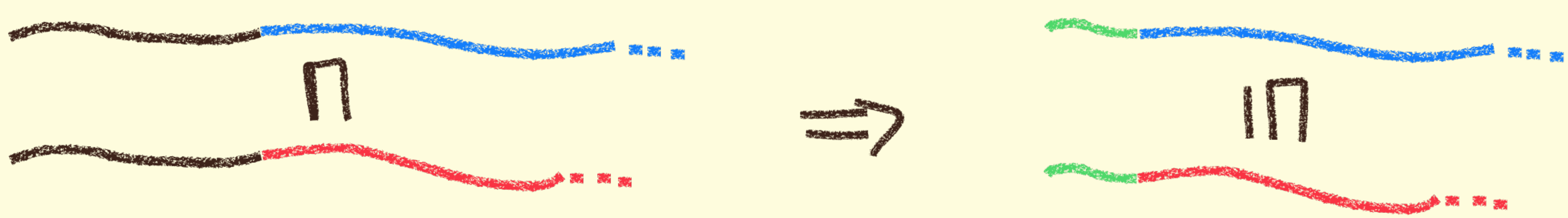

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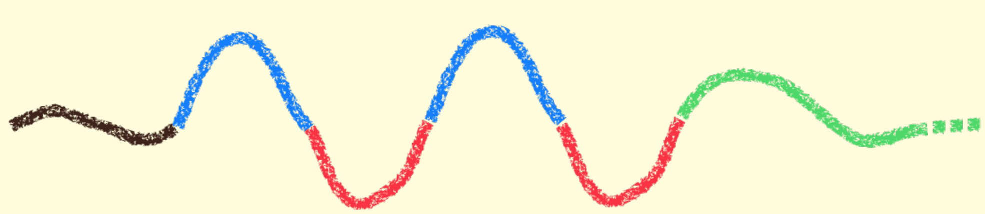

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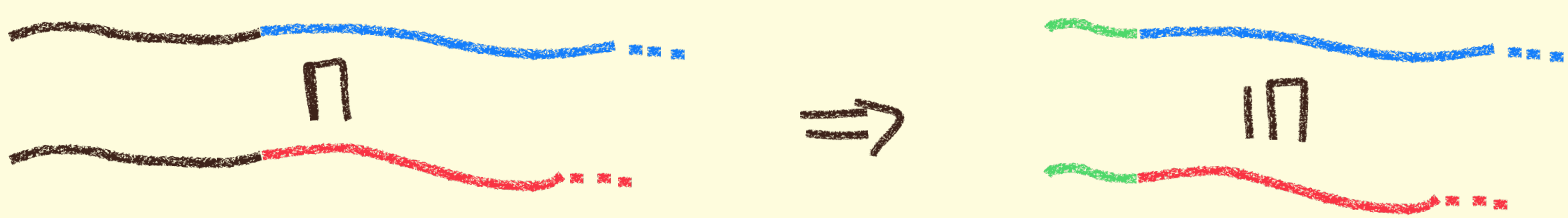

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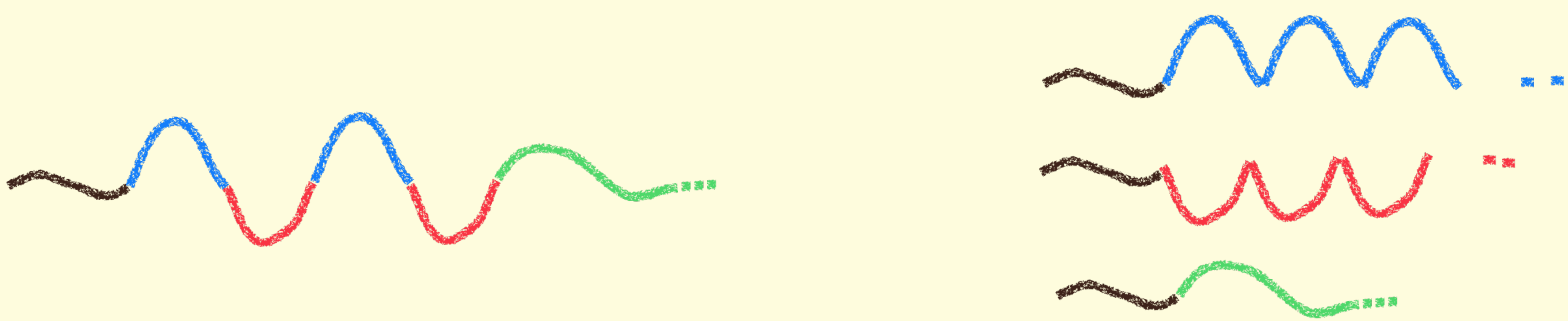

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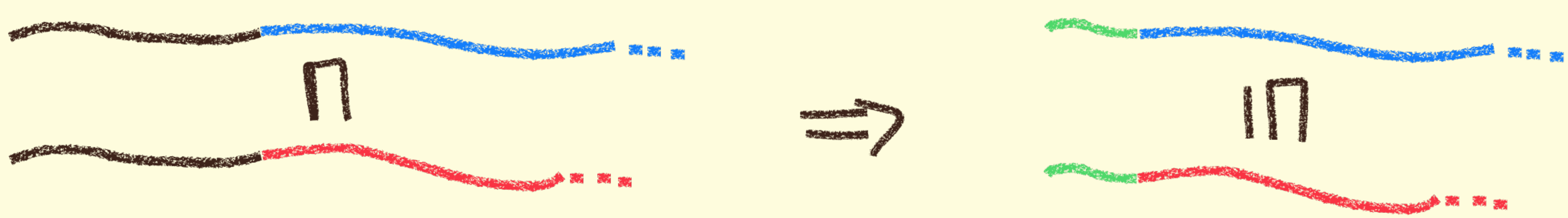

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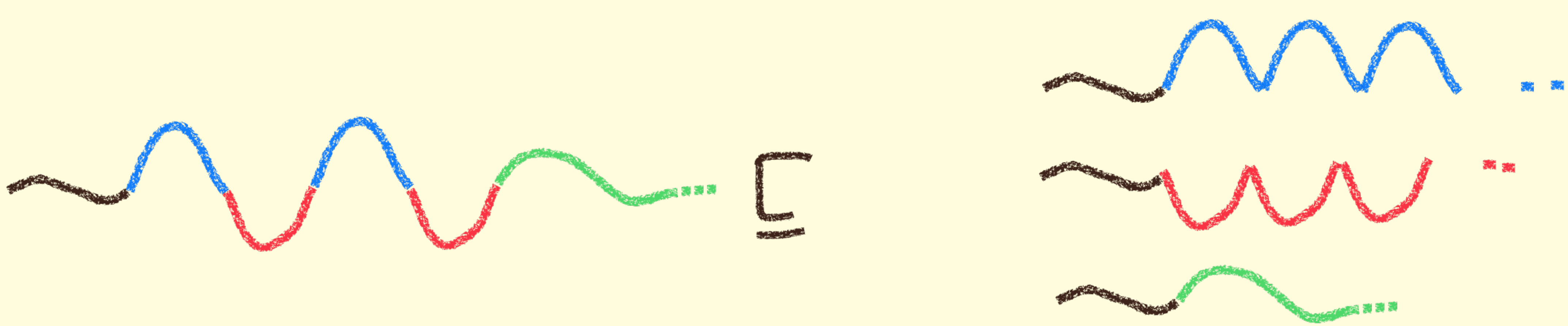

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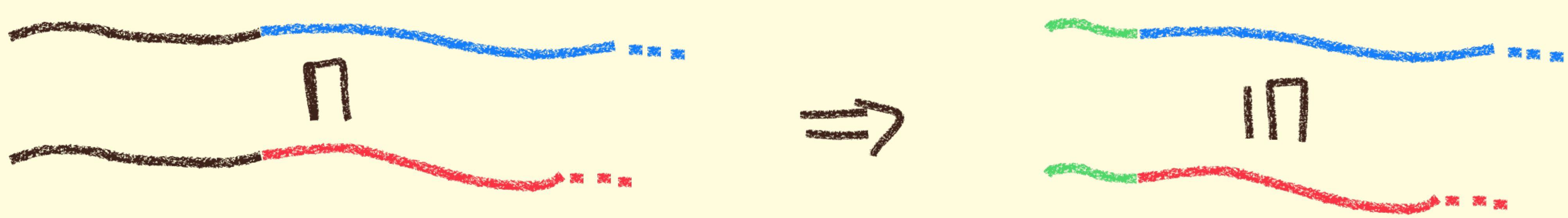

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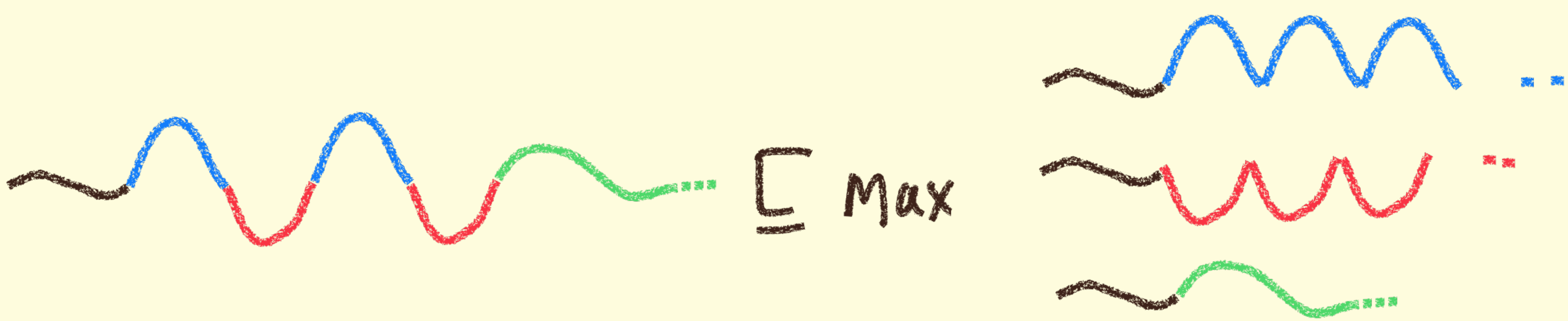

The Gimbert-Zielonka characterization for memory less determinacy (2)

[GZ05]

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Characterization - Two-player games

The two following assertions are equivalent :

1. All finite games have memoryless optimal strategies for both players
2. Both $\sqsubseteq$ and $\sqsubseteq^{-1}$ are monotone and selective

The Gimbert-Zielonka characterization for memory less determinacy (2)

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Characterization - Two-player games

The two following assertions are equivalent :

1. All finite games have memoryless optimal strategies for both players
2. Both $\sqsubseteq$ and $\sqsubseteq^{-1}$ are monotone and selective

Characterization - One-player games

The two following assertions are equivalent :

1. All finite P_1-games have (uniform) memoryless optimal strategies
2. $\sqsubseteq$ is monotone and selective

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Assume all P_1-games have optimal memoryless strategies.

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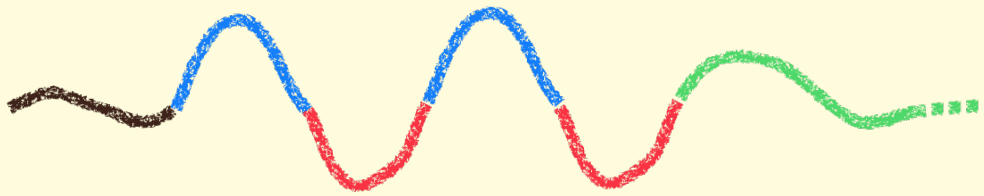

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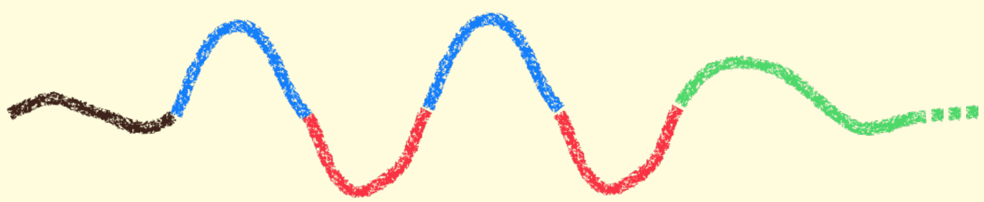

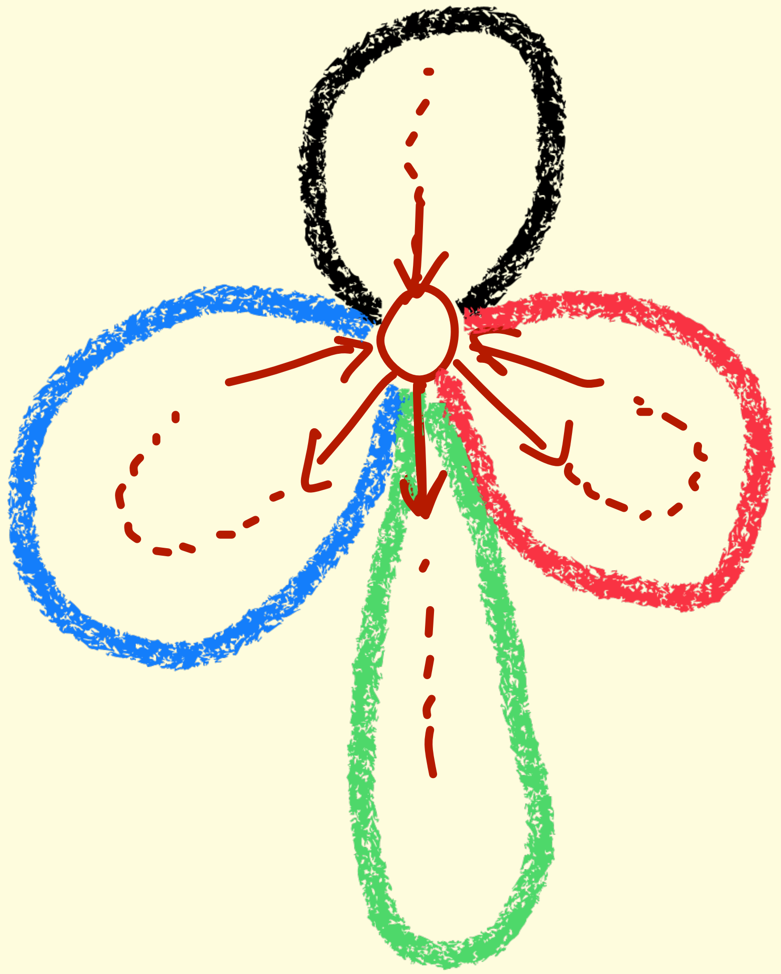

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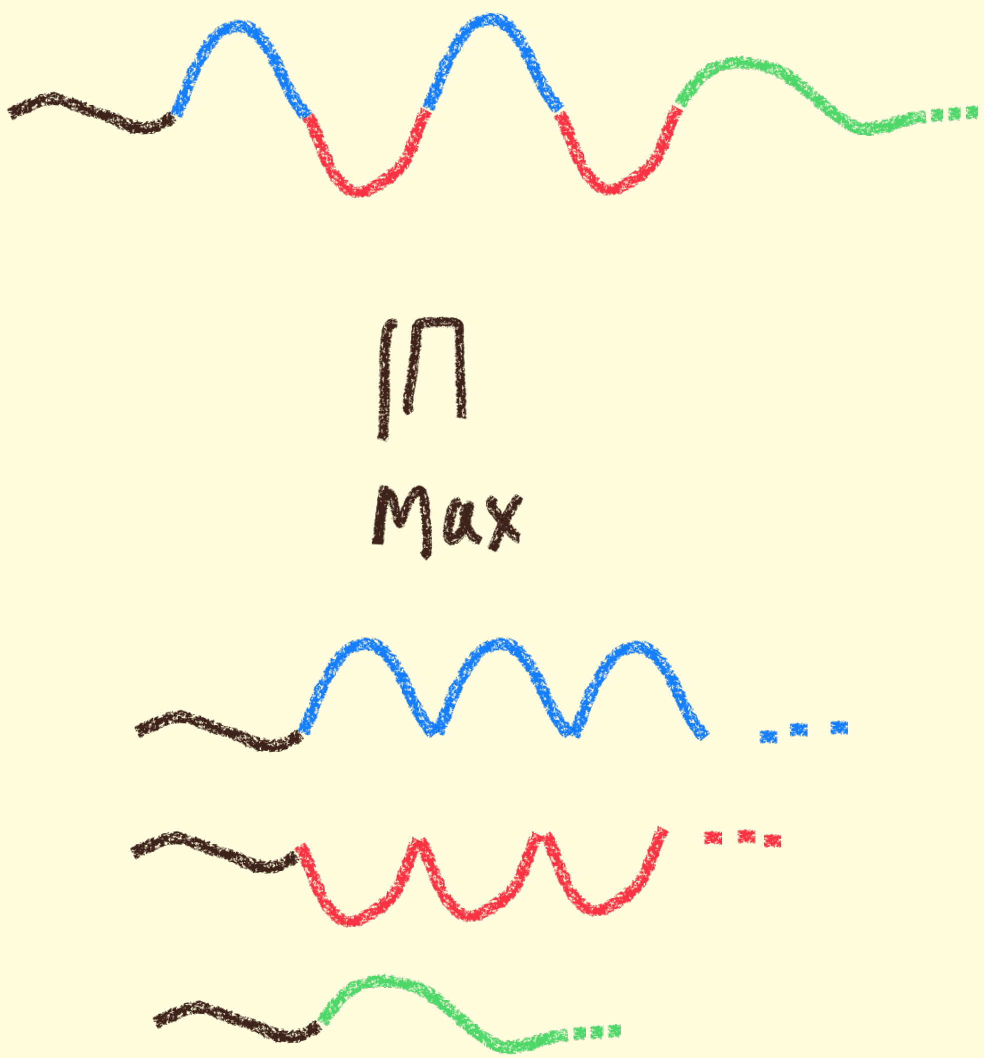

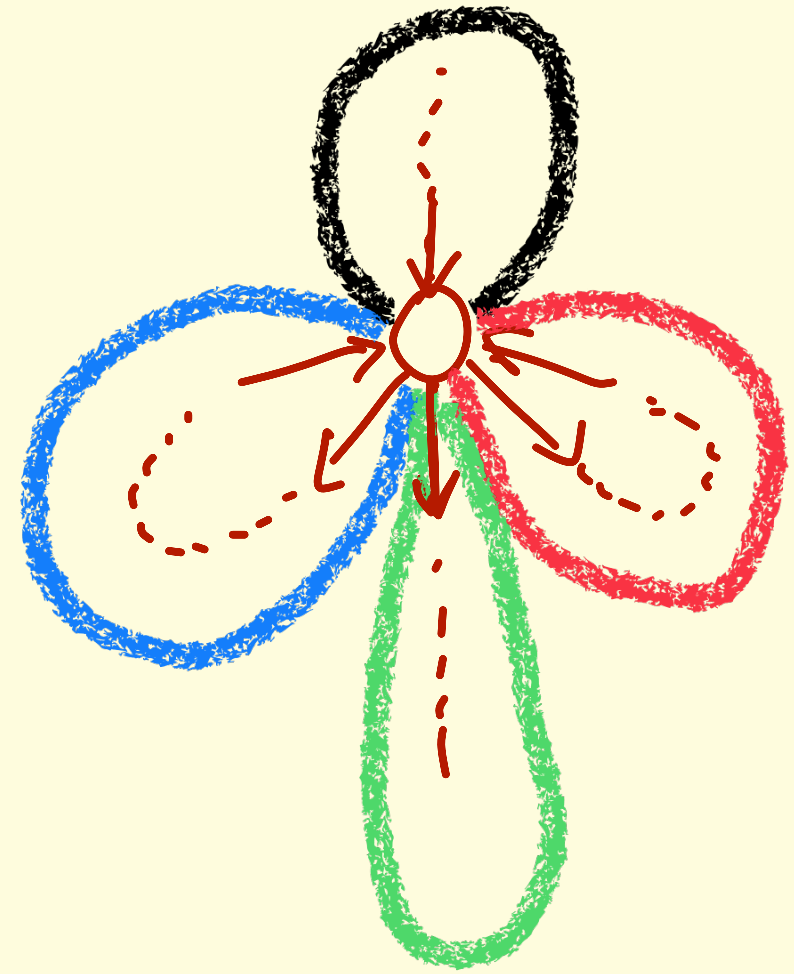

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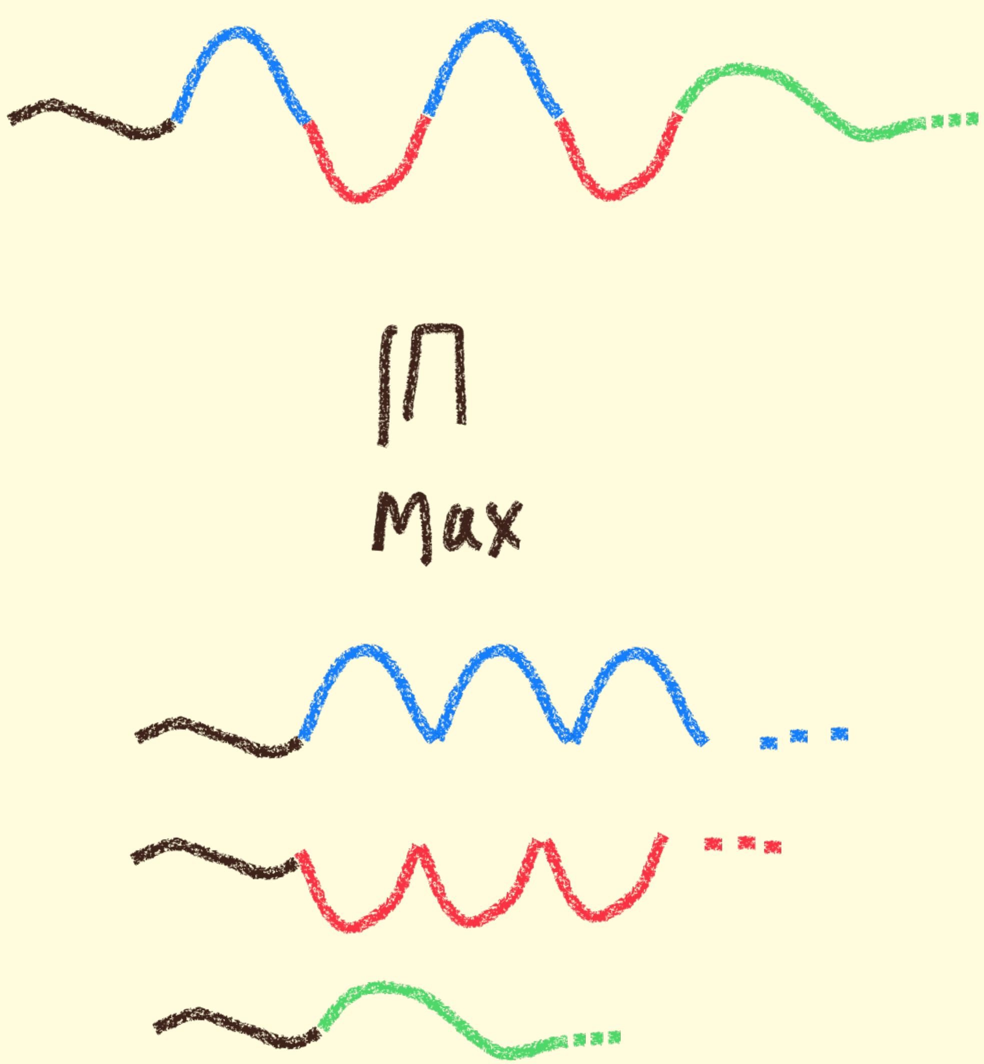
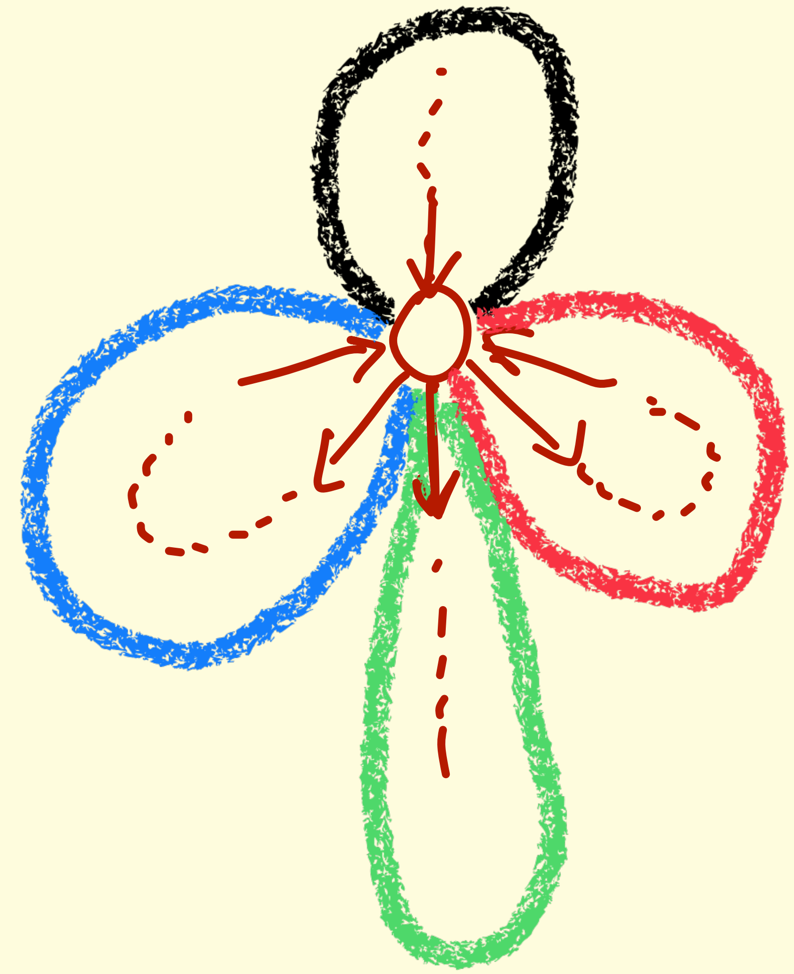


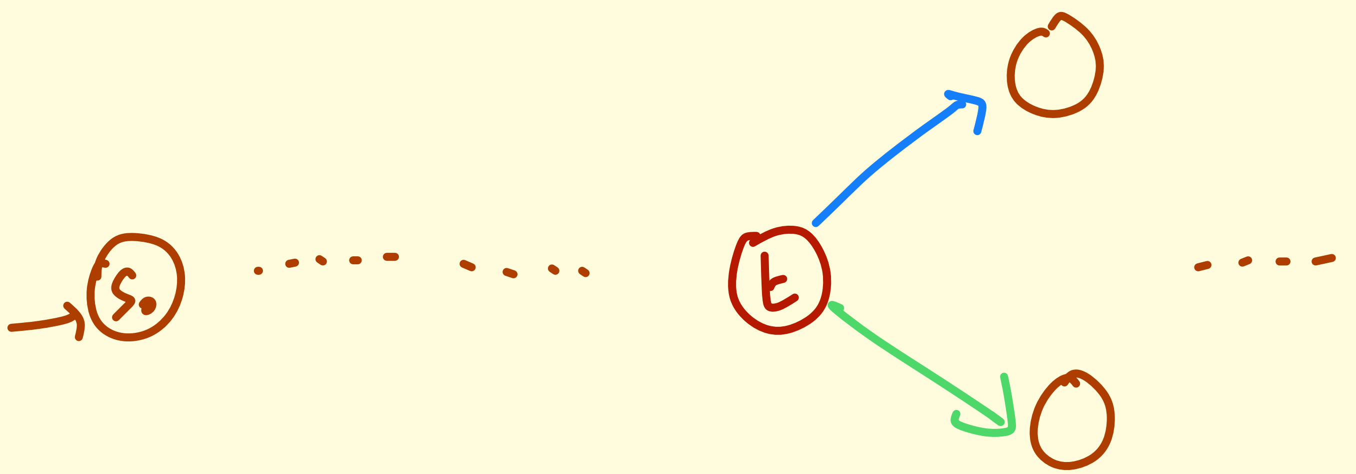


$\sqsubseteq$ is selective

Why? Proof hint (2)

Assume $\sqsubseteq$ is monotone and selective.

The case of one-player games



one best choice between $\nearrow$ and $\searrow$ (monotony)

+ no reason to swap at t (selectivity)

No memory required at t!

Applications

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- If in all finite one-player game for player P_i, P_i has uniform memoryless optimal strategies, then both players have memoryless optimal strategies in all finite two-player games.

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Discussion

- Easy to analyse the one-player case (graph analysis)
  - Mean-payoff, average-energy [BMRLL15]
- Allows to deduce properties in the two-player case

Discussion of examples

Examples

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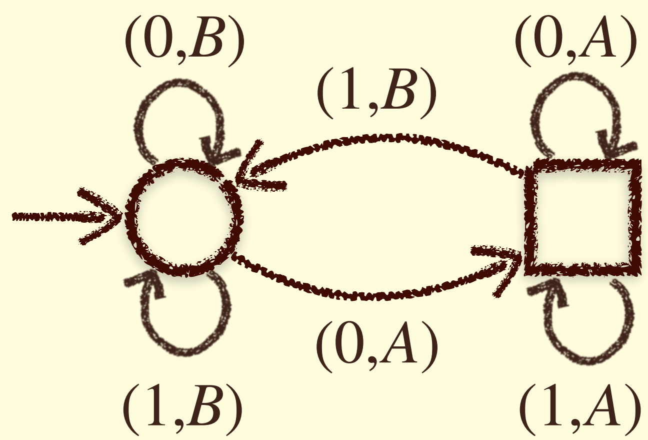

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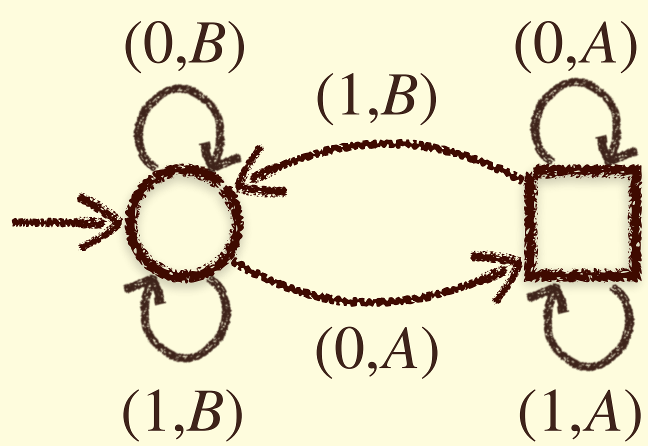

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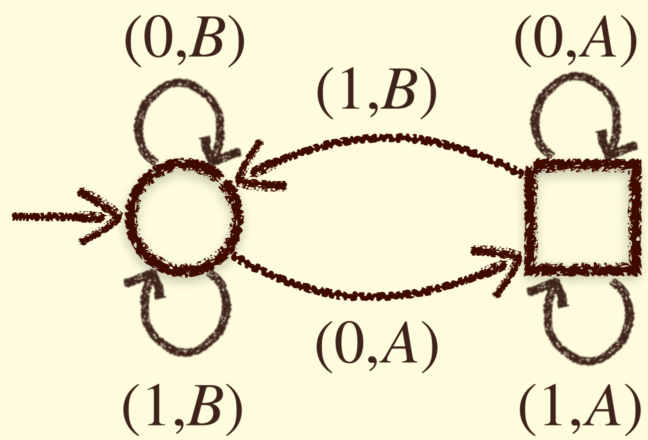

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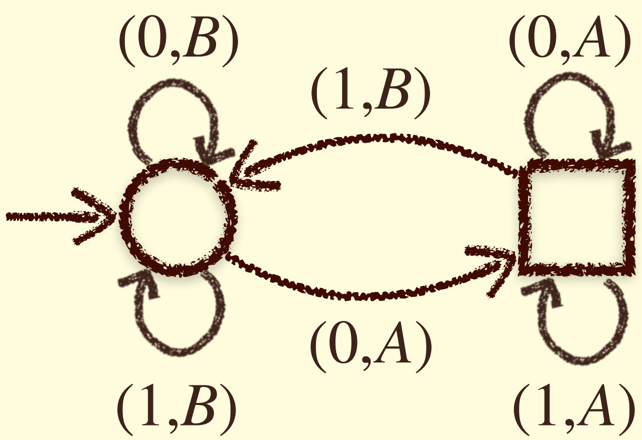

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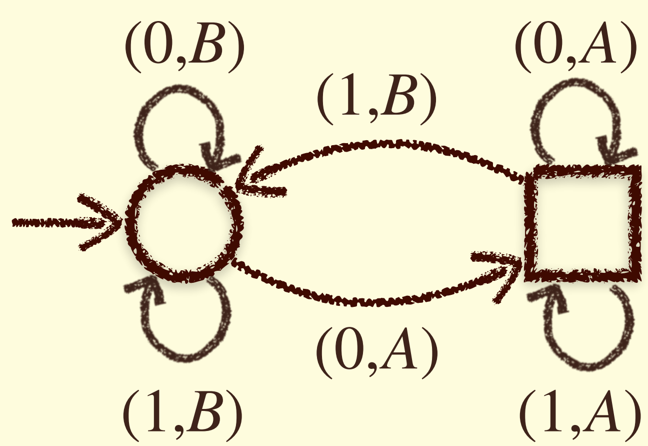

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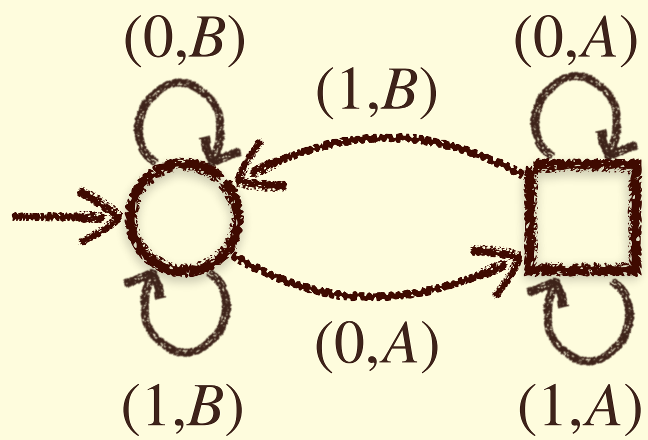

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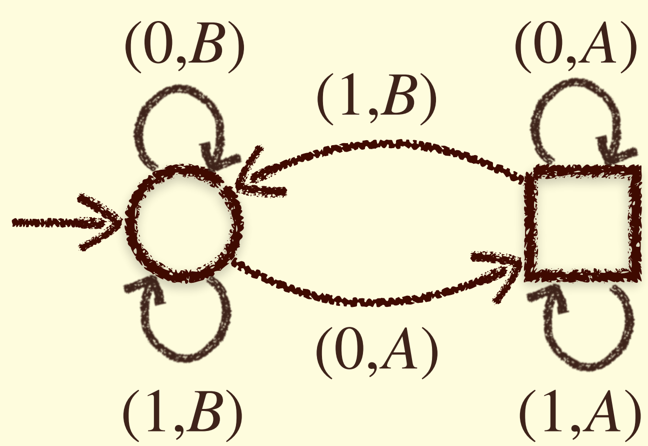

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- It requires infinite memory!

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Winning condition for P_1:

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If only $\sqsubseteq$ is monotone and selective, P_1 might not have a memoryless optimal strategy

Finite-memory strategies

We need memory!

Objectives/preference relations become more and more complex

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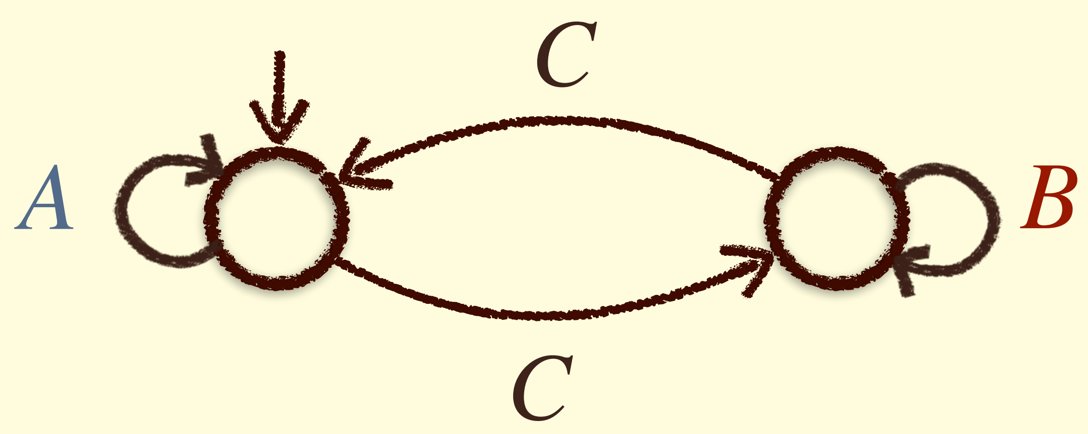

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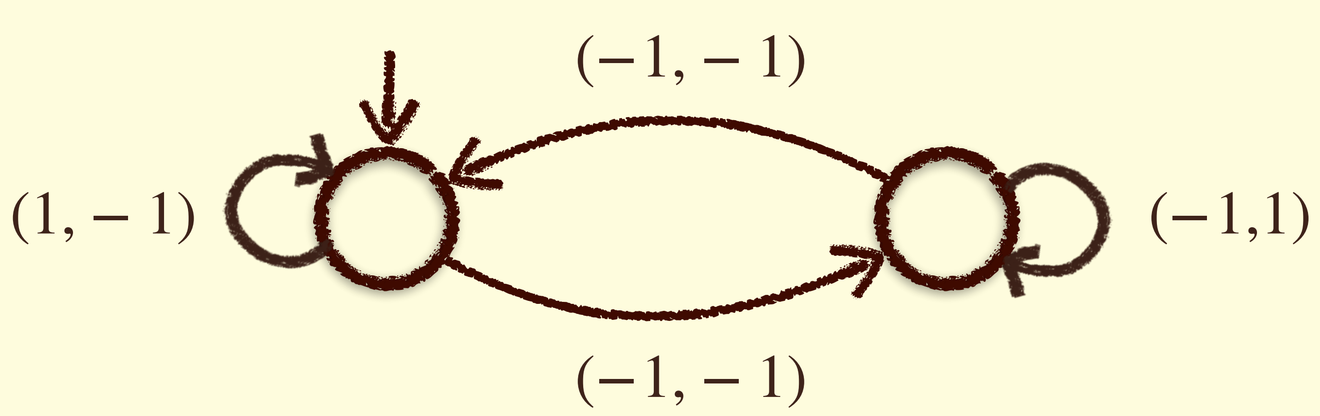
- $\text{Büchi}(A) \wedge \text{Büchi}(B)$ requires finite memory
- $MP_1 \geq 0 \wedge MP_2 \geq 0$ requires infinite memory



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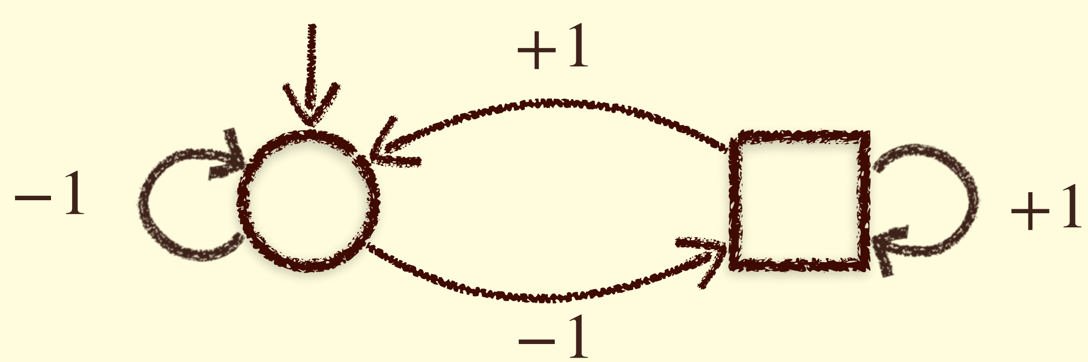

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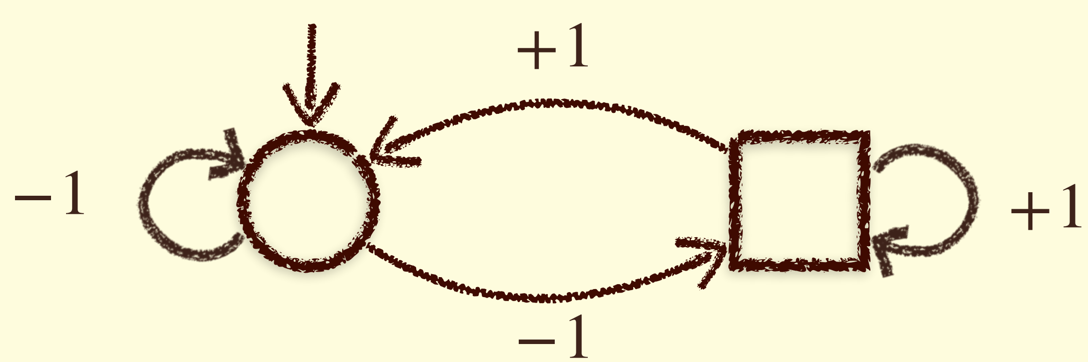

P_1 wins but uses infinite memory!

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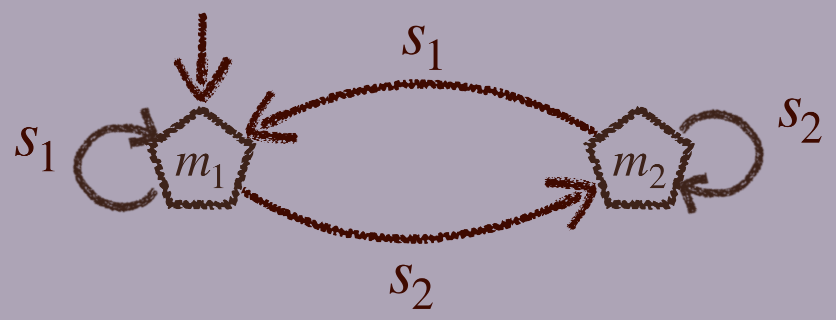

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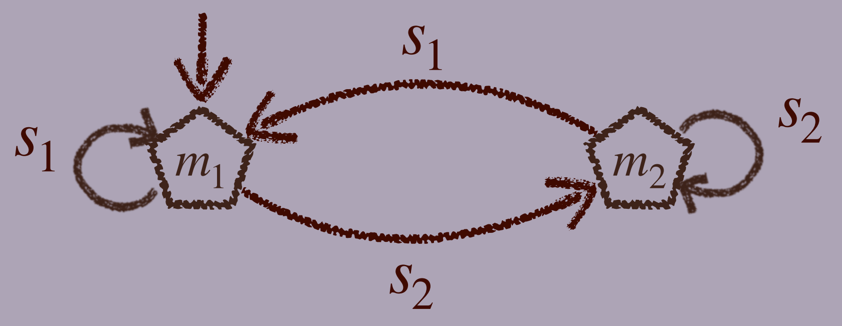

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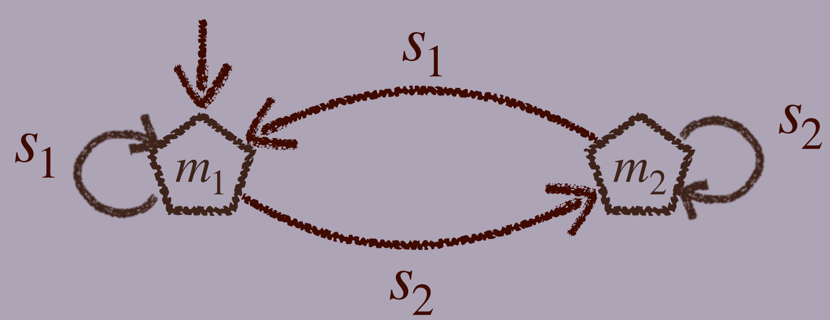

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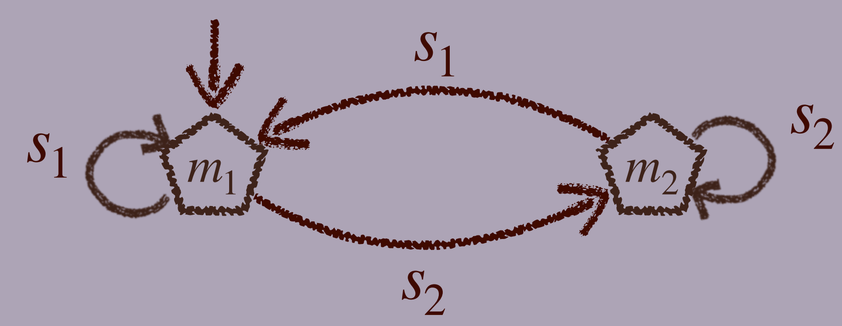

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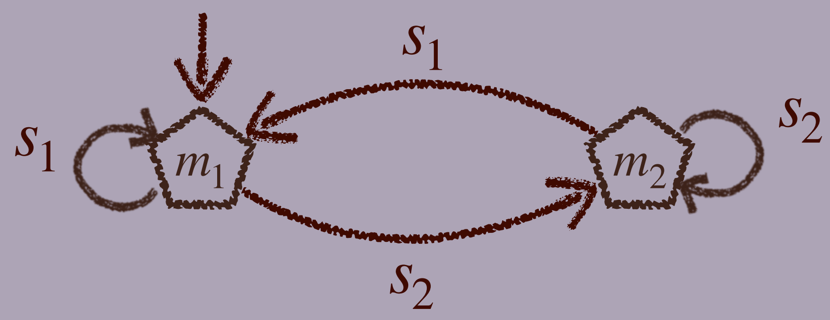

To have an abstract theorem…

- The memory mechanism should not speak about information specific to particular games, hence:
  - α_{upd} should not speak of states
  - α_{upd} can speak of colors
    (notion of « chromatic strategy » by Kopczynski)

Arena-independent memory management

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Memory skeleton

- $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ with $m_{\mathsf{init}} \in M$ and $\alpha_{\mathsf{upd}} : M \times C \to M$

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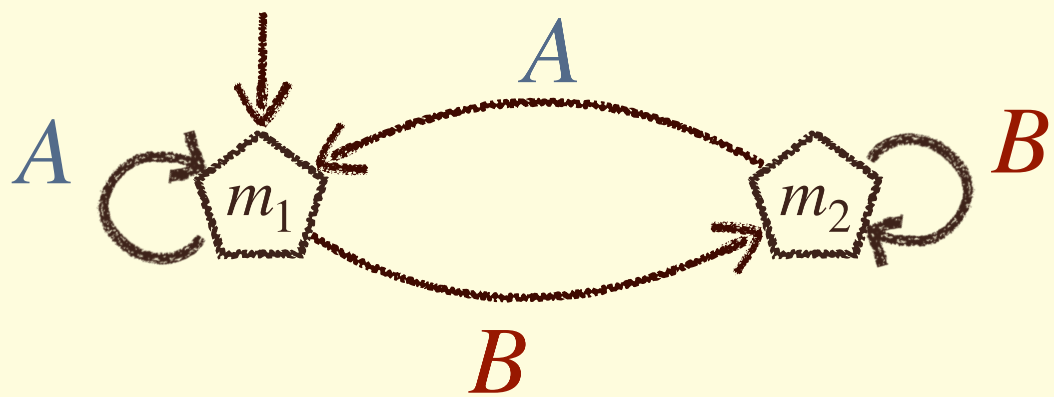

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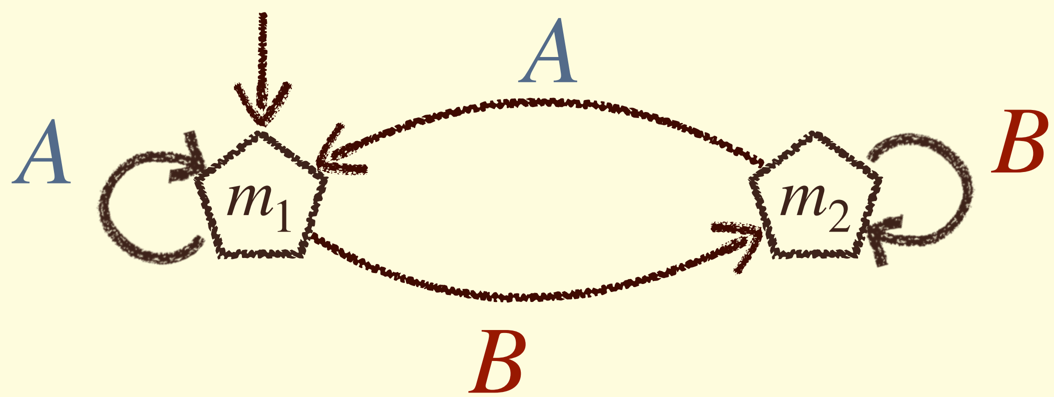

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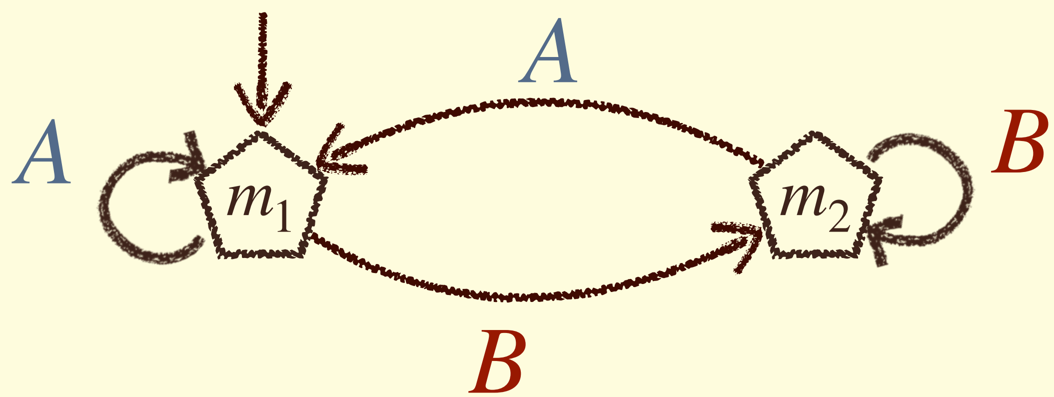

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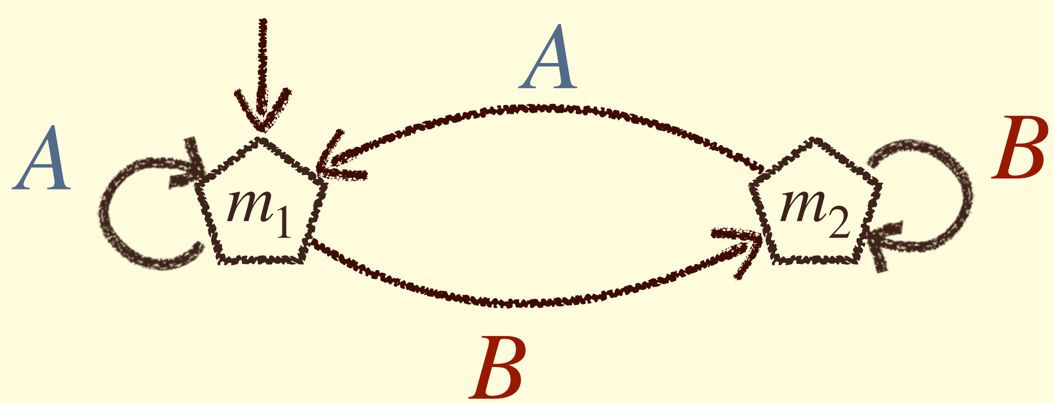

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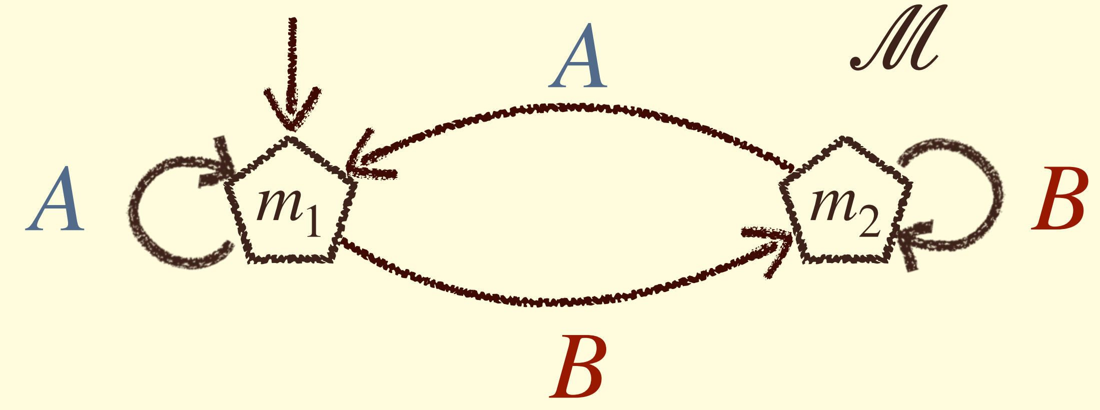
The above skeleton is sufficient for the winning condition

Büchi$(A) \wedge$Büchi(B)

Example



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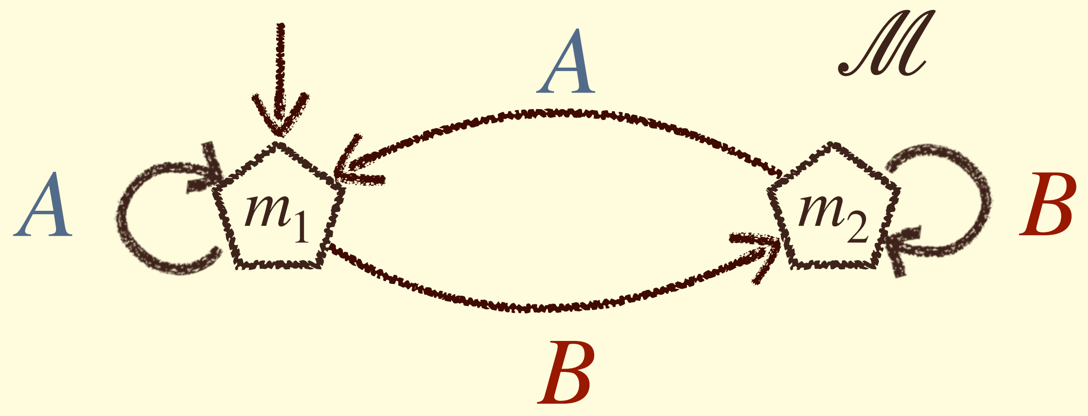
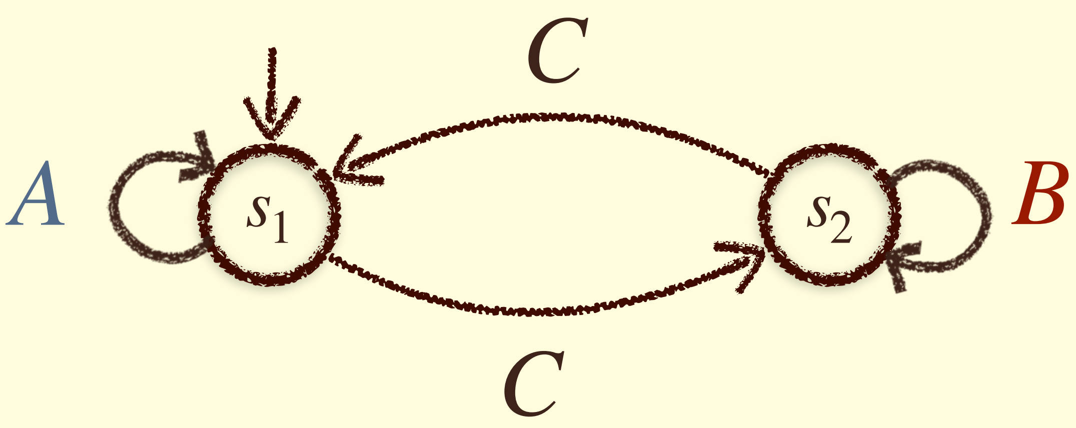


Game arena $\mathscr{A}$:



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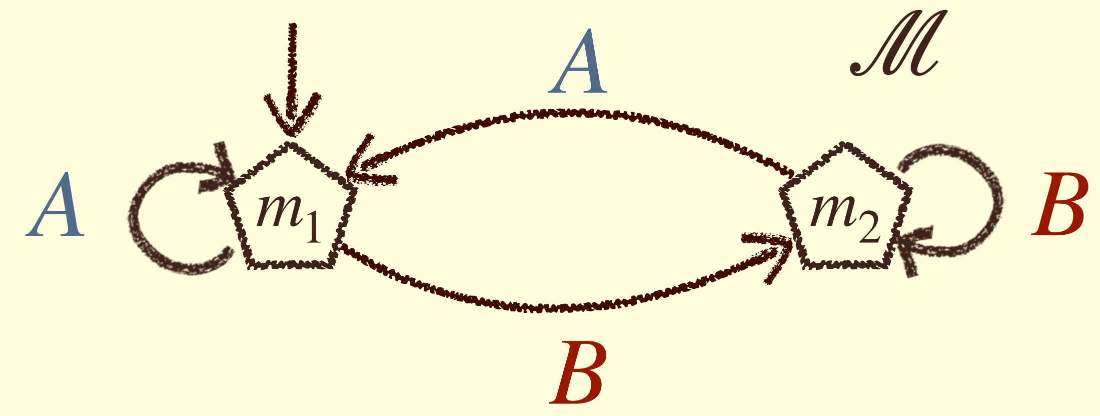

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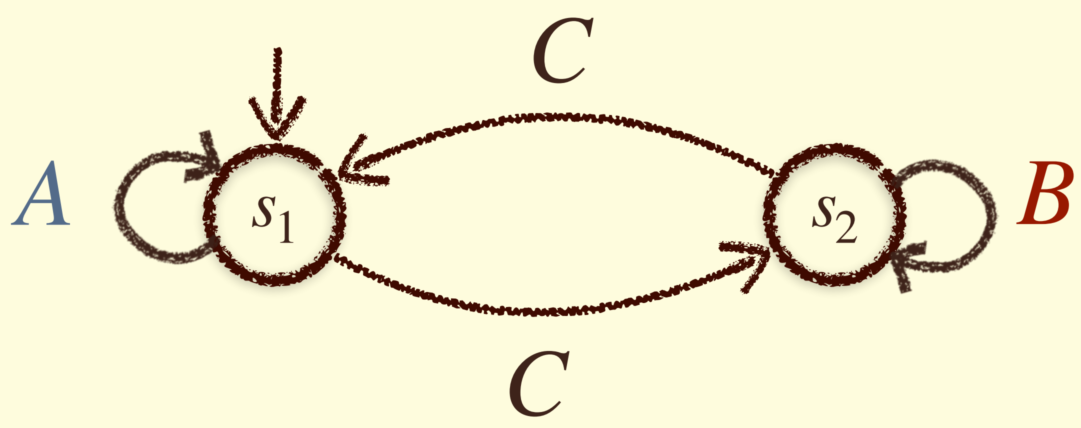

$$(s_1, m_1) \mapsto (s_1, s_2)$$
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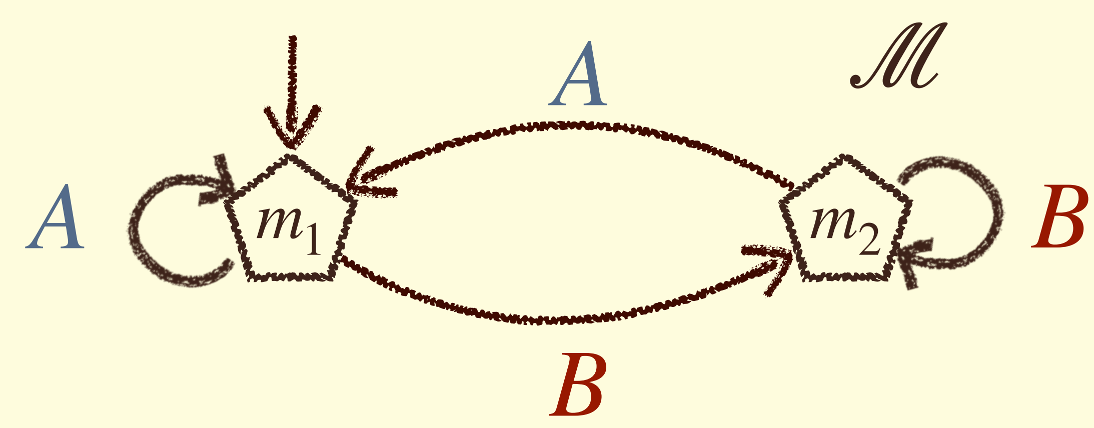

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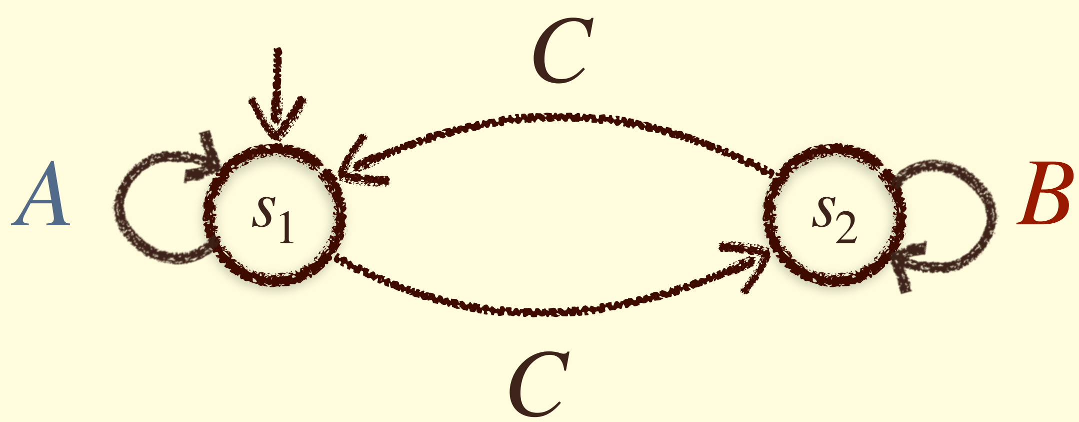

$$
\begin{aligned}
(s_1, m_1) &\mapsto (s_1, s_2)\\
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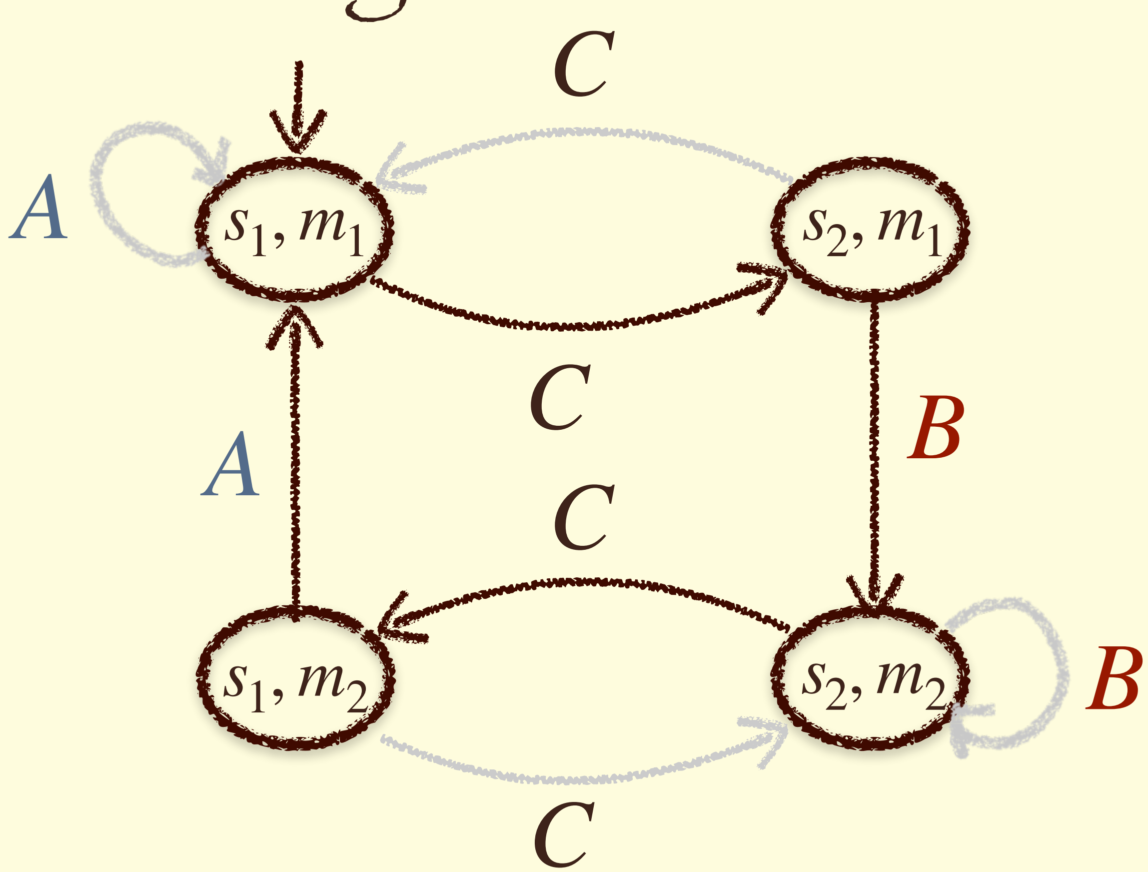
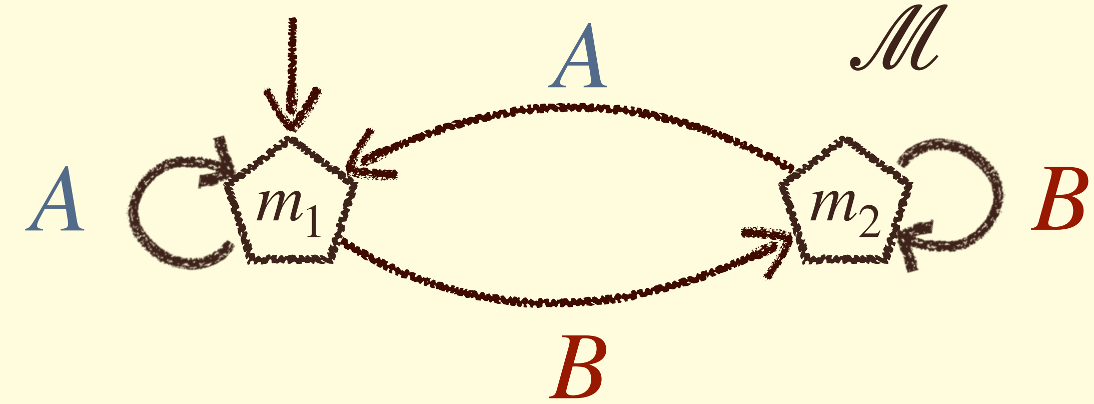
\end{aligned}
$$

Product game $\mathcal{A} \ltimes \mathcal{M}$:



Example



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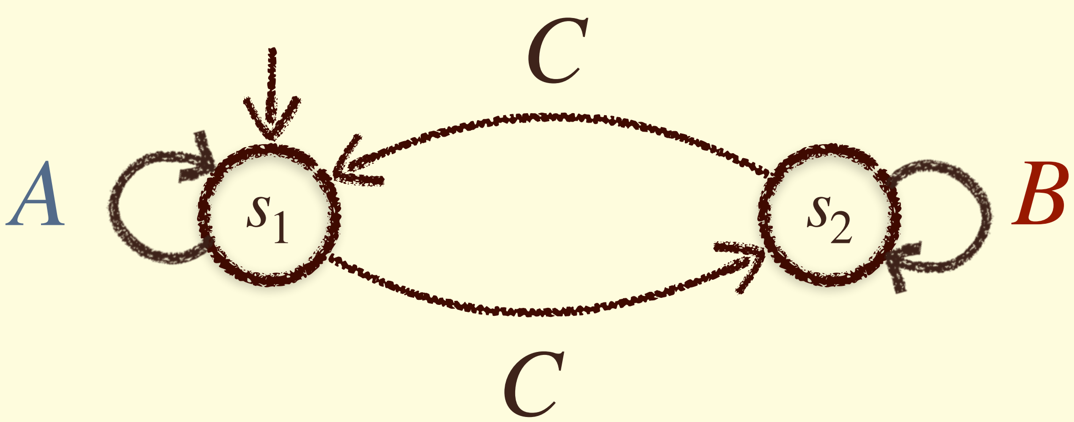

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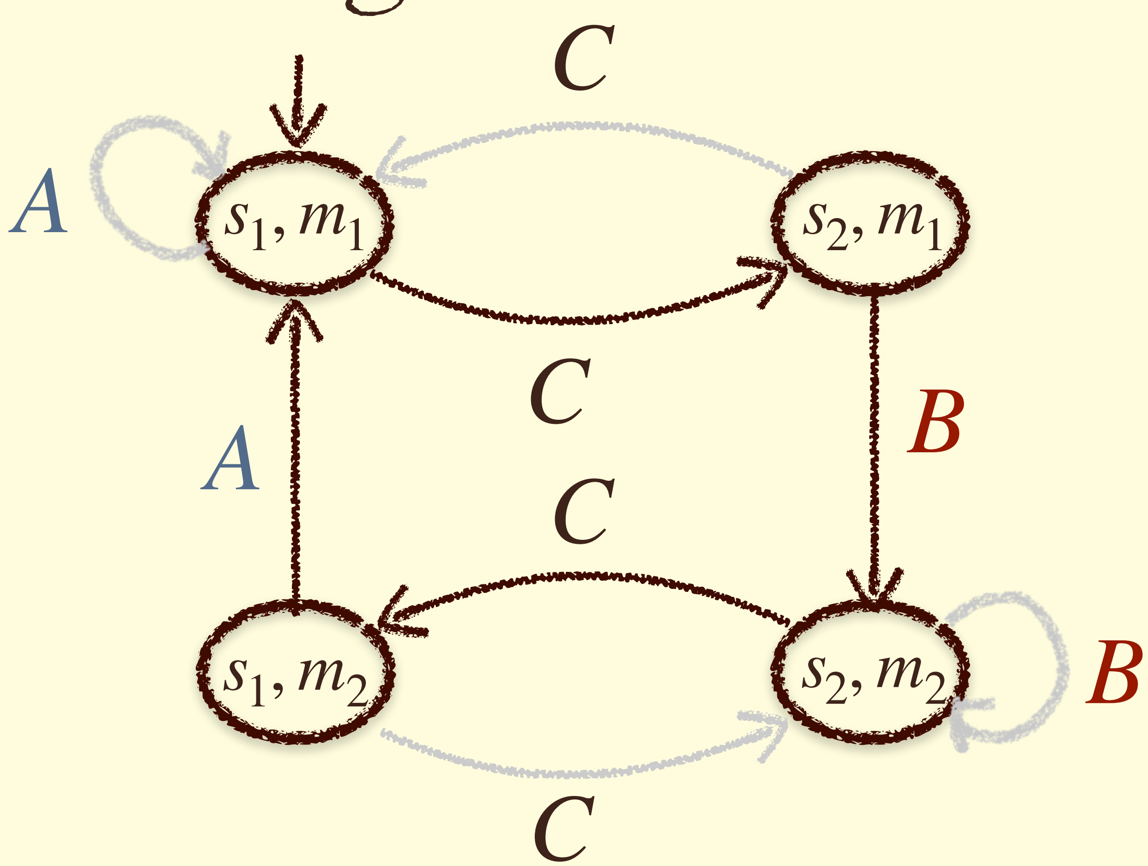

- One can however not apply the [GZ05] result to product games!

Memory-dependent monotony and selectivity

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Let $\sqsubseteq$ be a preference relation and $\mathcal{M}$ a memory skeleton.

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It is said :

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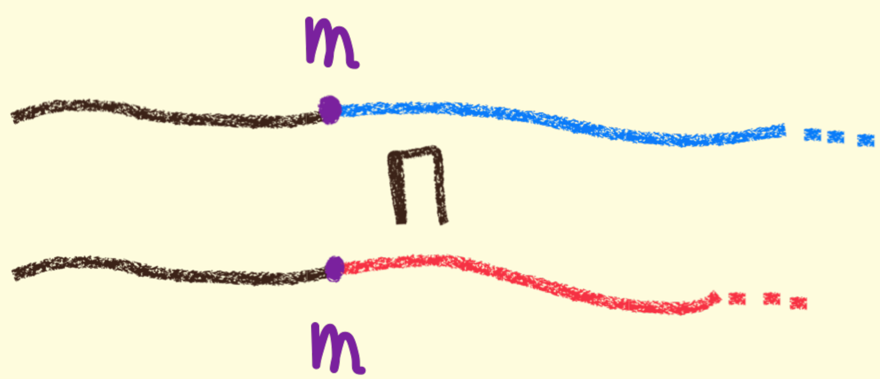

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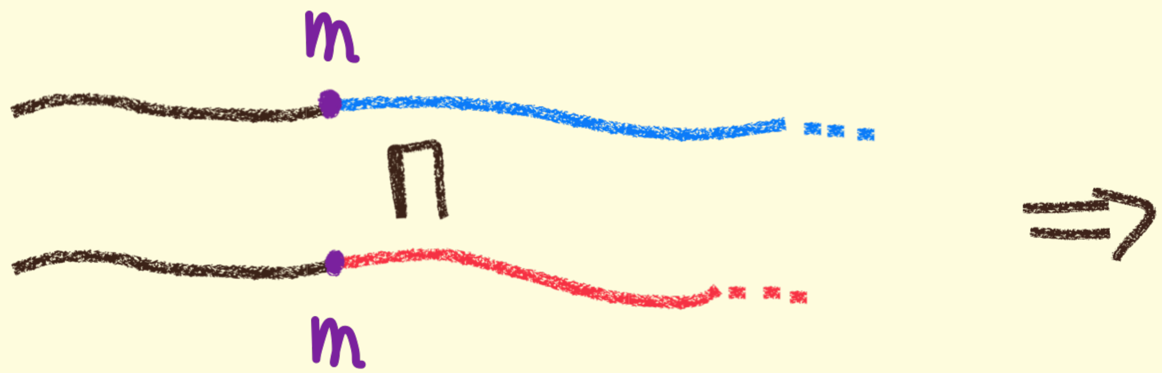

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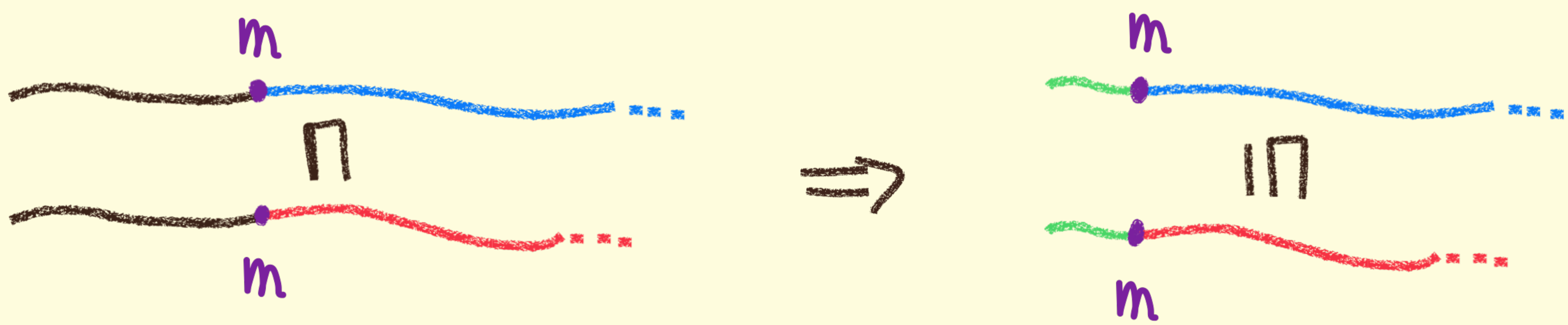

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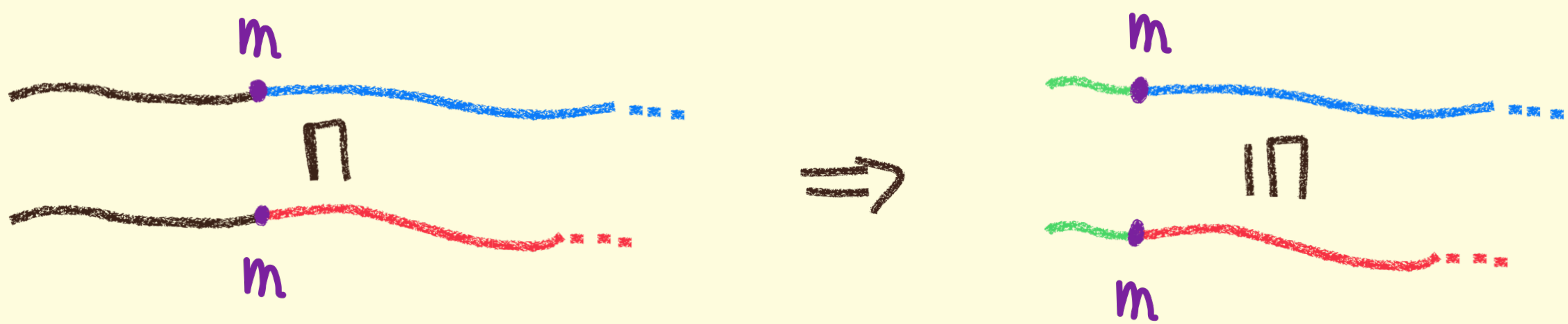

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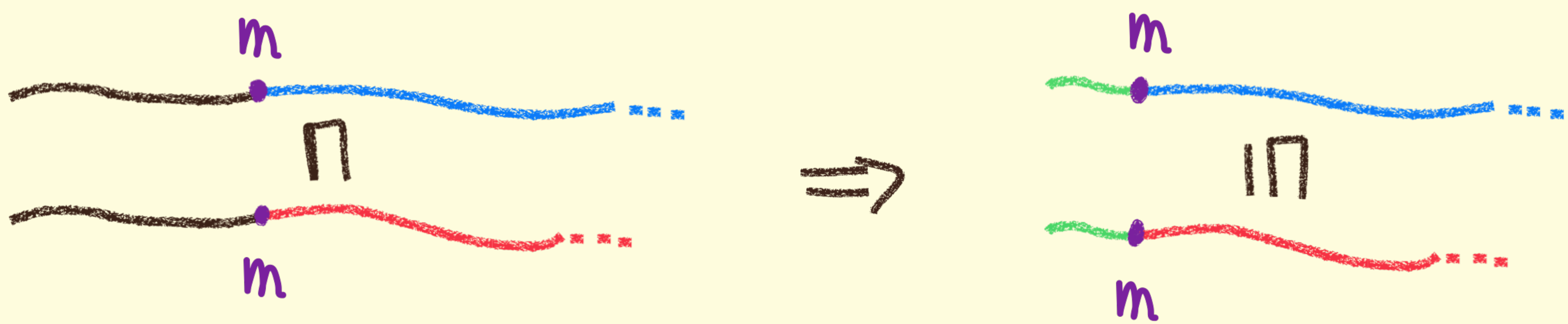

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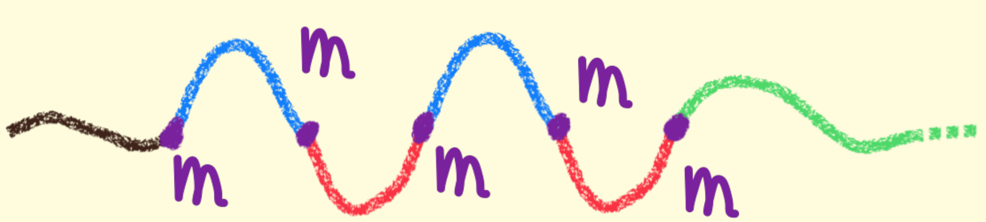

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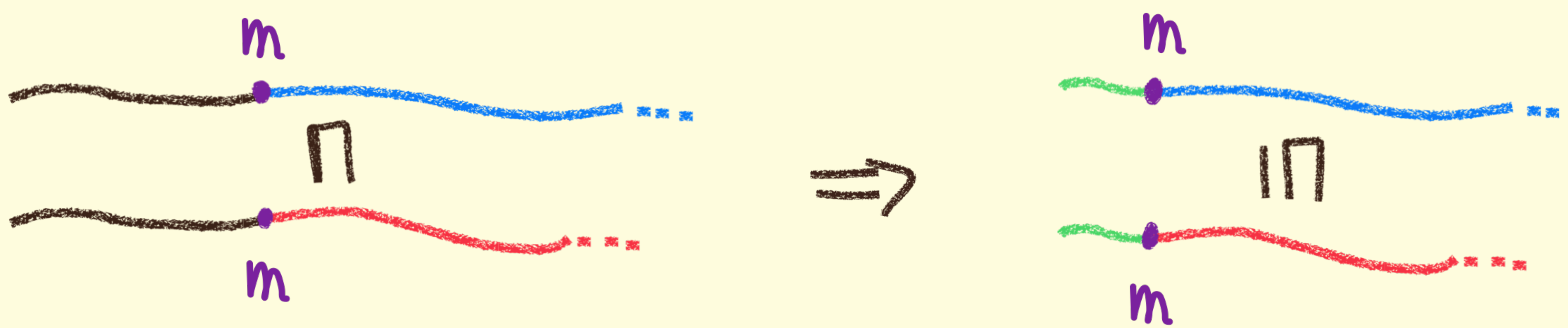

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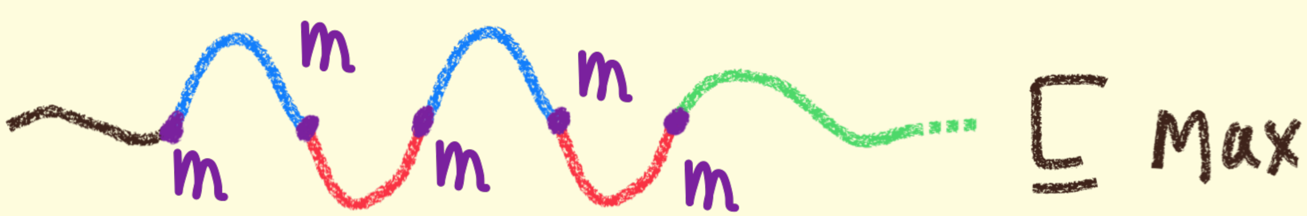

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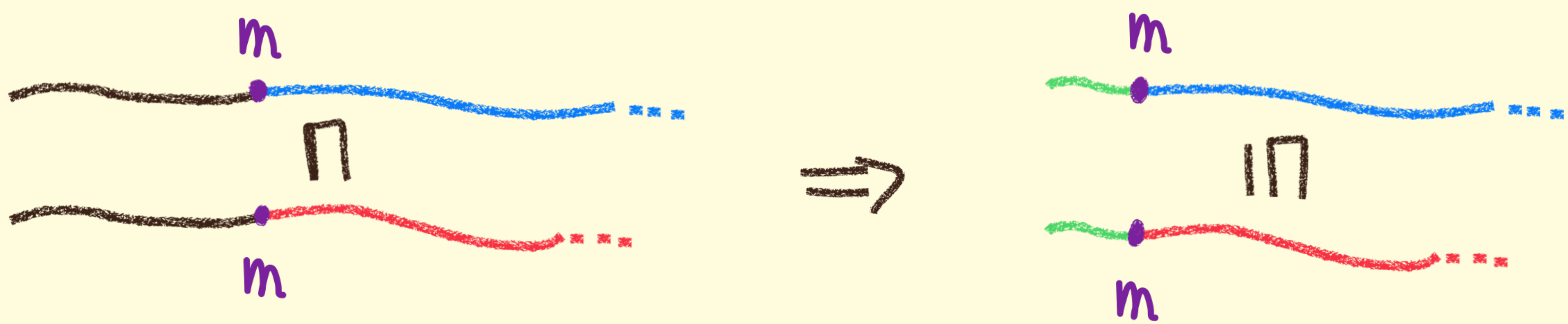

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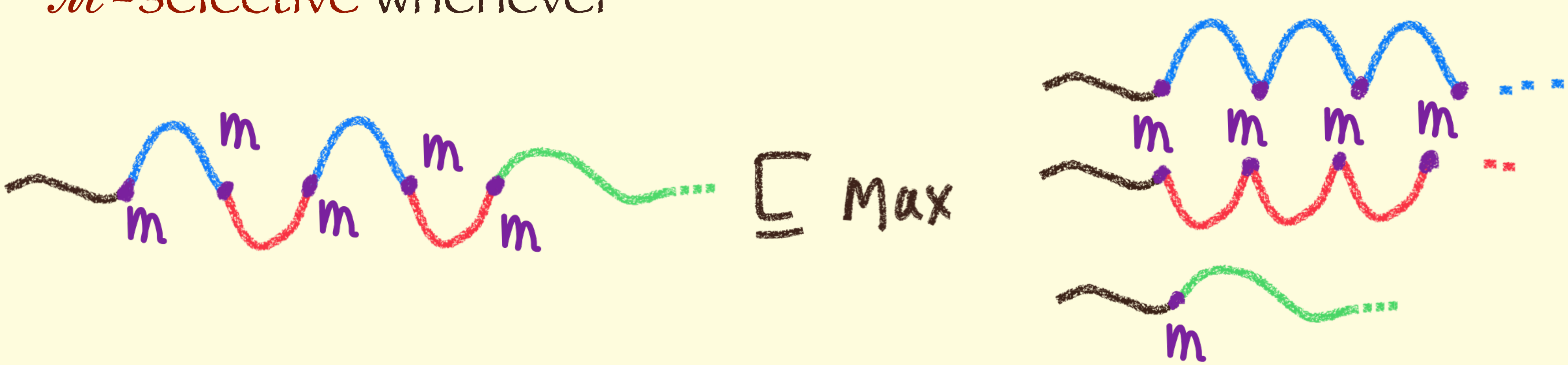

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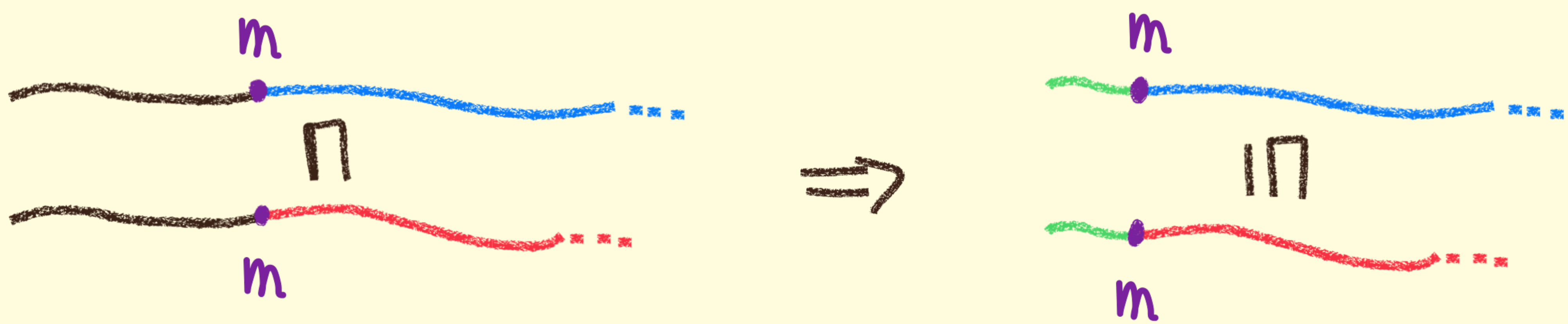

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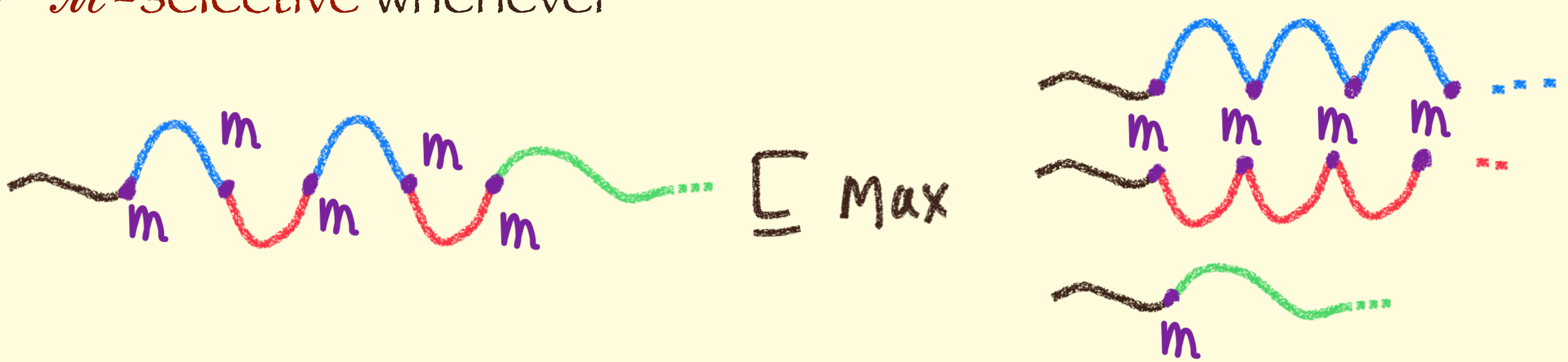

We look at how $\mathcal{M}$ classifies prefixes and cycles

Formal definitions of $\mathcal{M}$-monotony and $\mathcal{M}$-selectivity

Definition ($\mathcal{M}$-monotony)

Let $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is *$\mathcal{M}$-monotone* if for all $m \in M$, for all $K_1, K_2 \in \mathcal{R}(C)$,

$$\exists\, w \in L_{m_{\mathsf{init}},m},\ [wK_1] \sqsubset [wK_2] \implies \forall\, w' \in L_{m_{\mathsf{init}},m},\ [w'K_1] \sqsubseteq [w'K_2].$$

Definition ($\mathcal{M}$-selectivity)

Let $\mathcal{M} = (M, m_{\mathsf{init}}, \alpha_{\mathsf{upd}})$ be a memory skeleton. A preference relation $\sqsubseteq$ is *$\mathcal{M}$-selective* if for all $w \in C^*$, $m = \widehat{\alpha_{\mathsf{upd}}}(m_{\mathsf{init}}, w)$, for all $K_1, K_2 \in \mathcal{R}(C)$ such that $K_1, K_2 \subseteq L_{m,m}$, for all $K_3 \in \mathcal{R}(C)$,

$$[w(K_1 \cup K_2)^* K_3] \sqsubseteq [wK_1^*] \cup [wK_2^*] \cup [wK_3].$$

Our characterization for $\mathscr{M}$-determinacy

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Characterization - Two-player games

Our characterization for $\mathcal{M}$-determinacy

Characterization - Two-player games

The two following assertions are equivalent :

1. All finite games have optimal $\mathcal{M}$-strategies for both players
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➡ We recover [GZ05] with $\mathcal{M} = \mathcal{M}_{\text{triv}}$

Applications

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Transfer/Lifting theorem

- If in all finite one-player game for player P_i, P_i has optimal $\mathcal{M}_i$-strategies, then both players have optimal $\mathcal{M}_1 \times \mathcal{M}_2$-strategies in all finite two-player games.

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Very powerful and extremely useful in practice!

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Subclasses of games

- If both $\sqsubseteq$ and $\sqsubseteq^{-1}$ are $\mathcal{M}$-monotone and $\mathcal{M}$-selective, then both players have optimal memoryless strategies in all $\mathcal{M}$-covered games.

Memory-covered arenas

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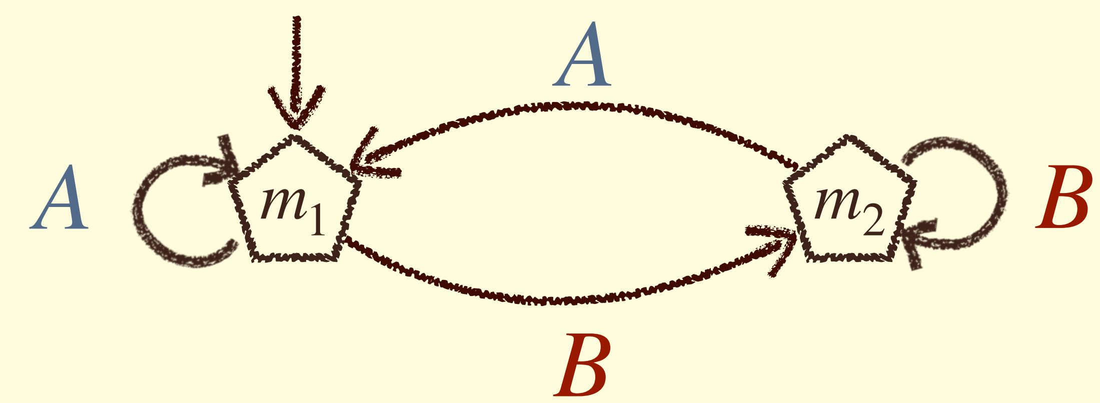

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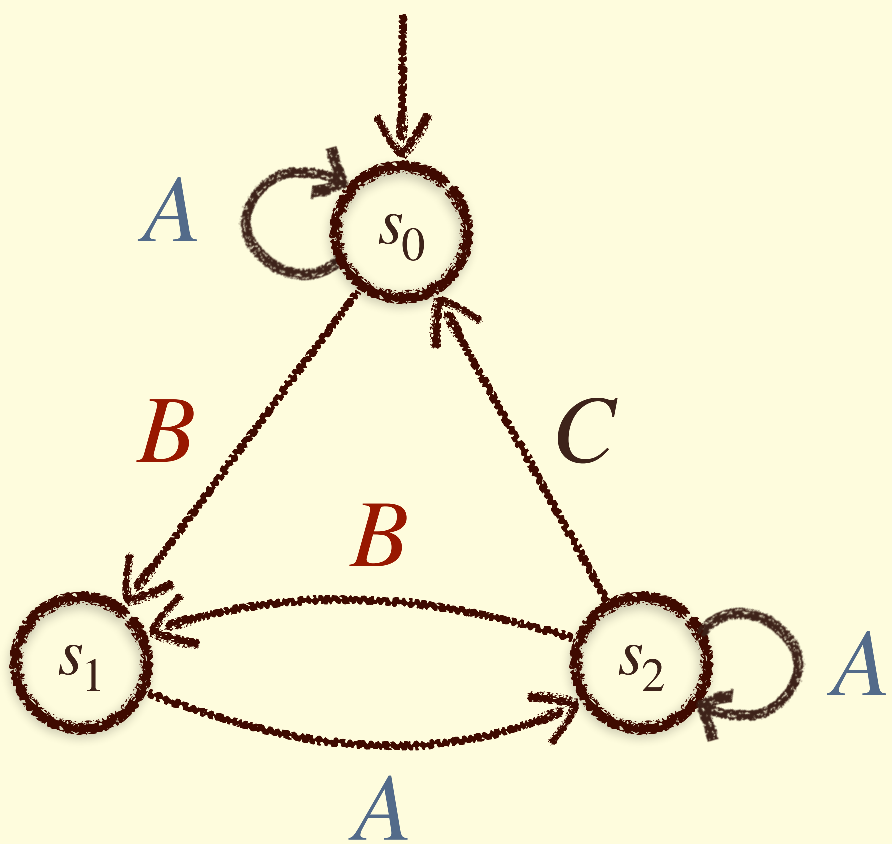

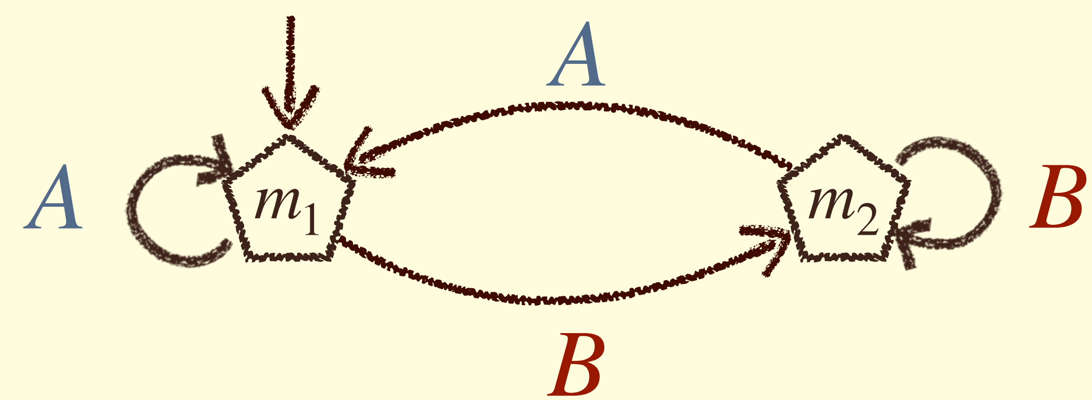

Example of application

$\sqsubseteq$ defined by a conjunction of reachability Reach(●) $\wedge$ Reach(●)

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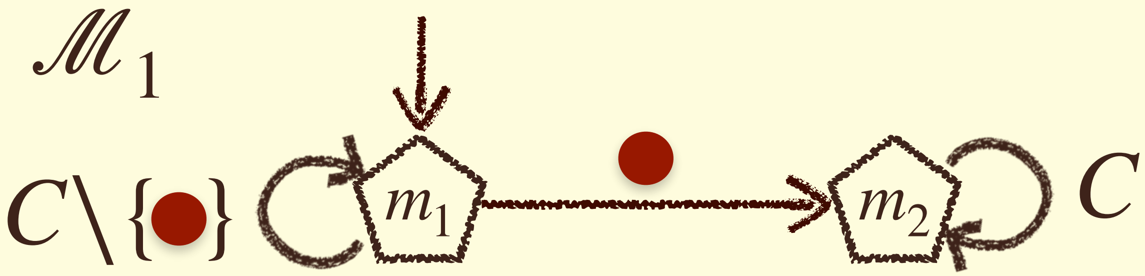

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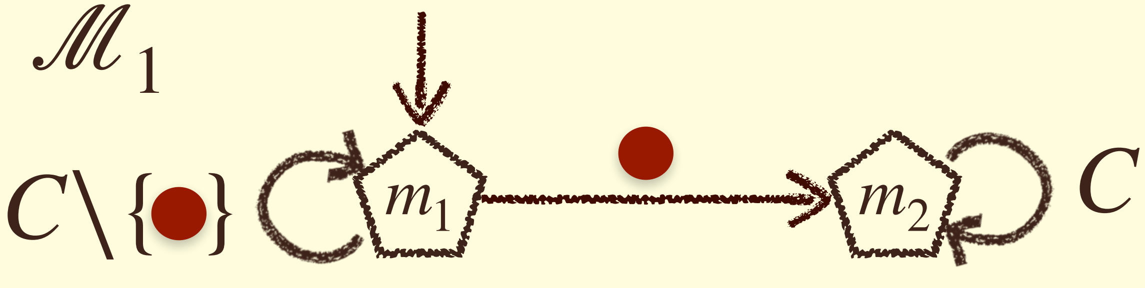

$\sqsubseteq$ is $\mathscr{M}_1$-monotone,

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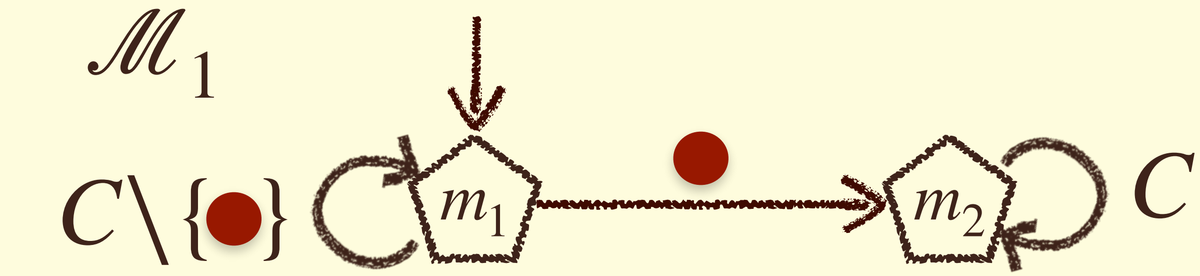

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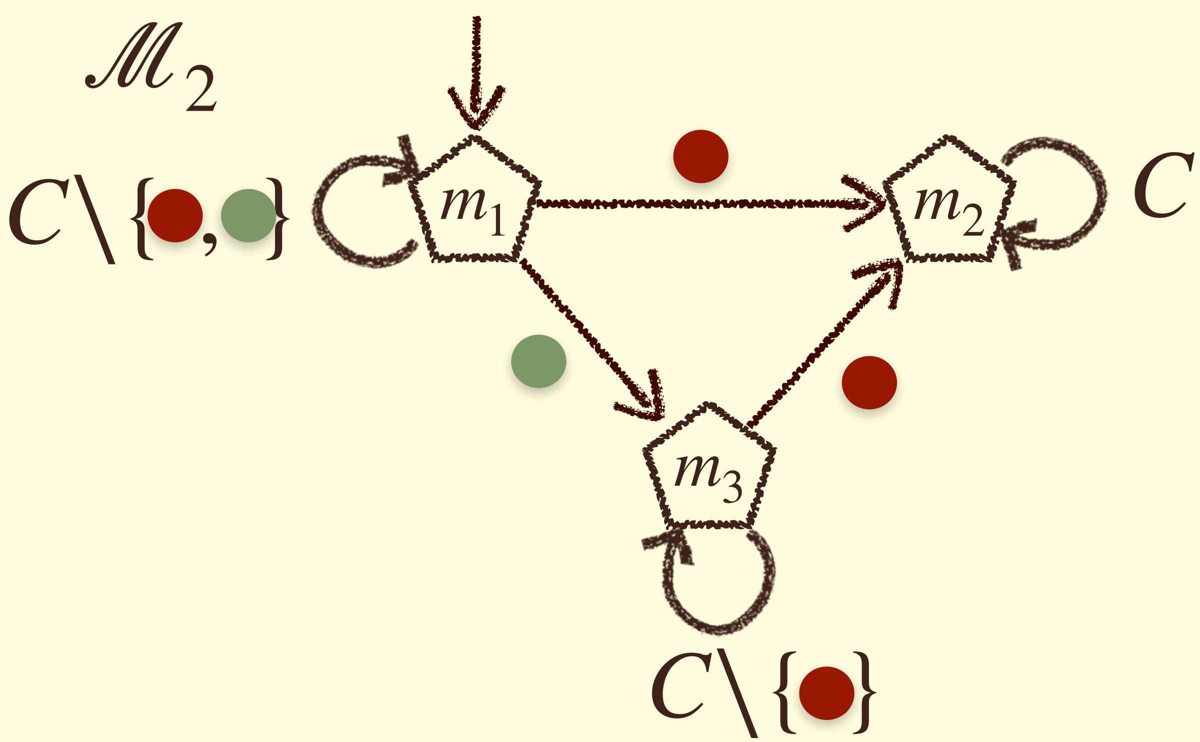

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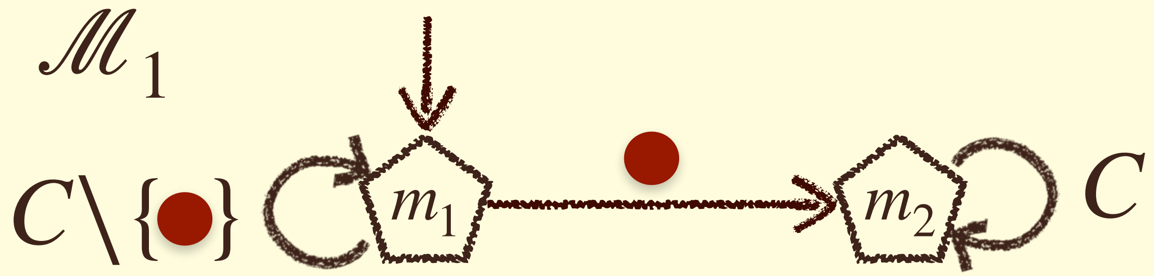

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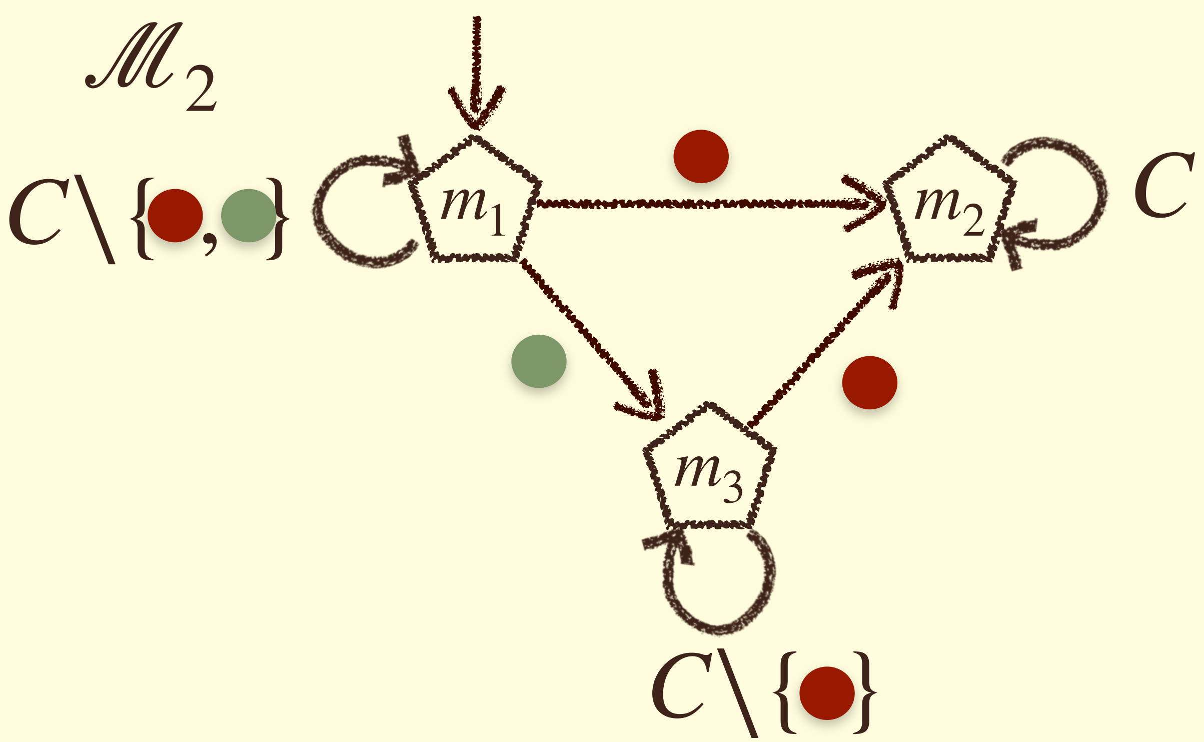

$\sqsubseteq$ is $\mathcal{M}_2$-selective

Example of application

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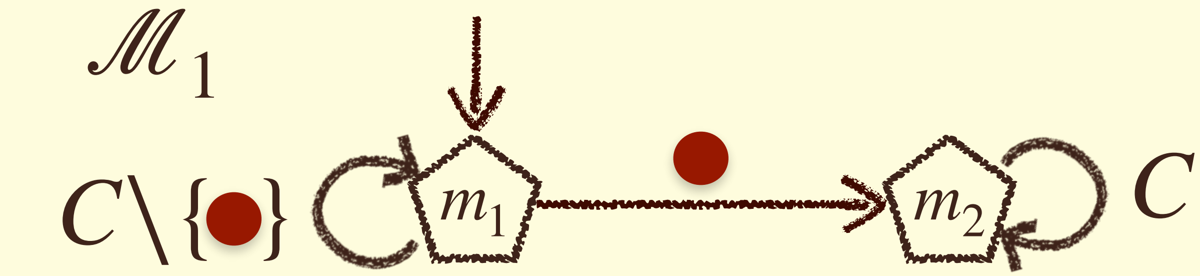

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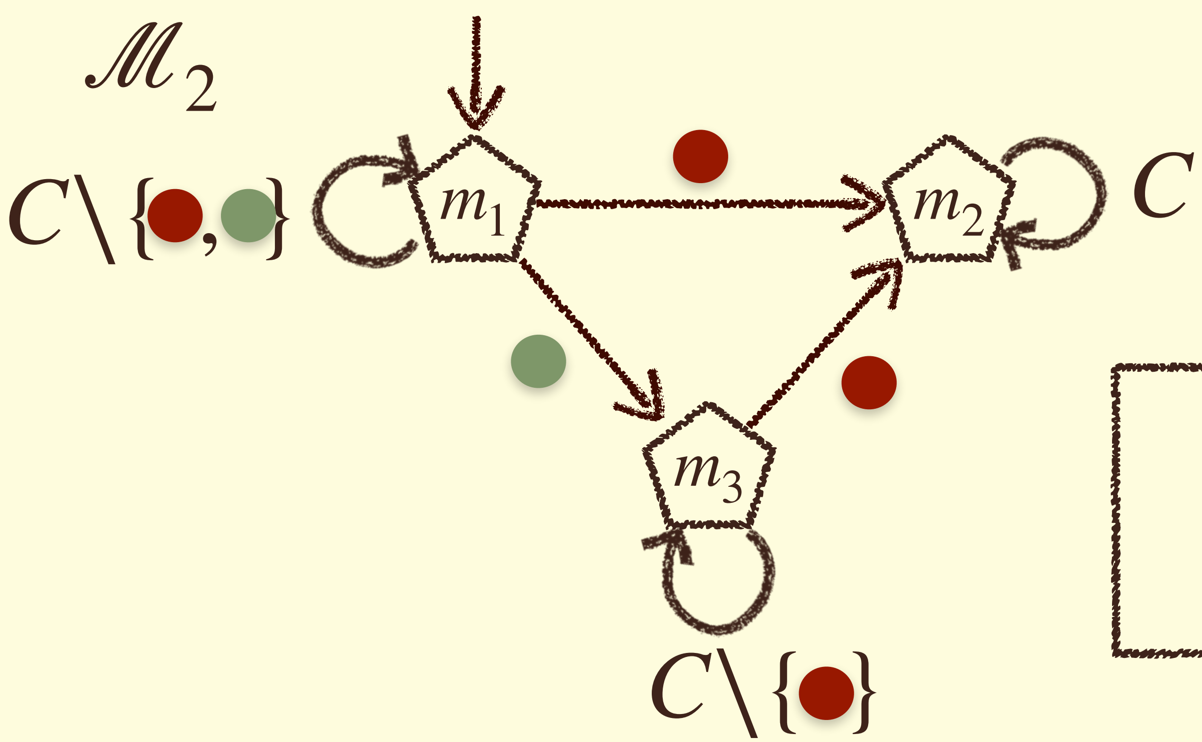

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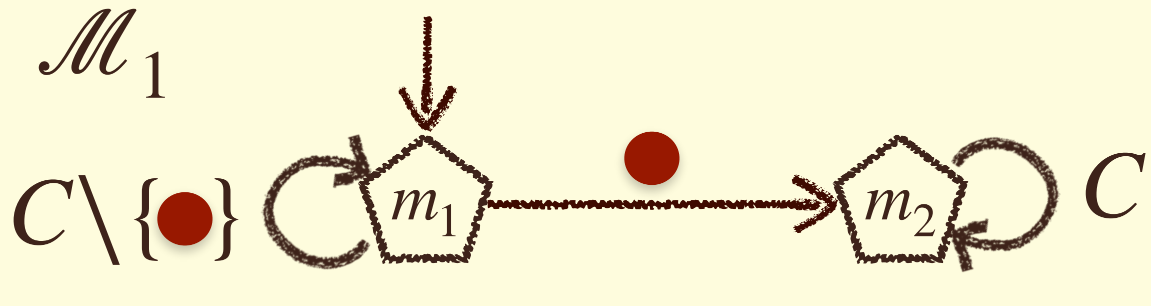

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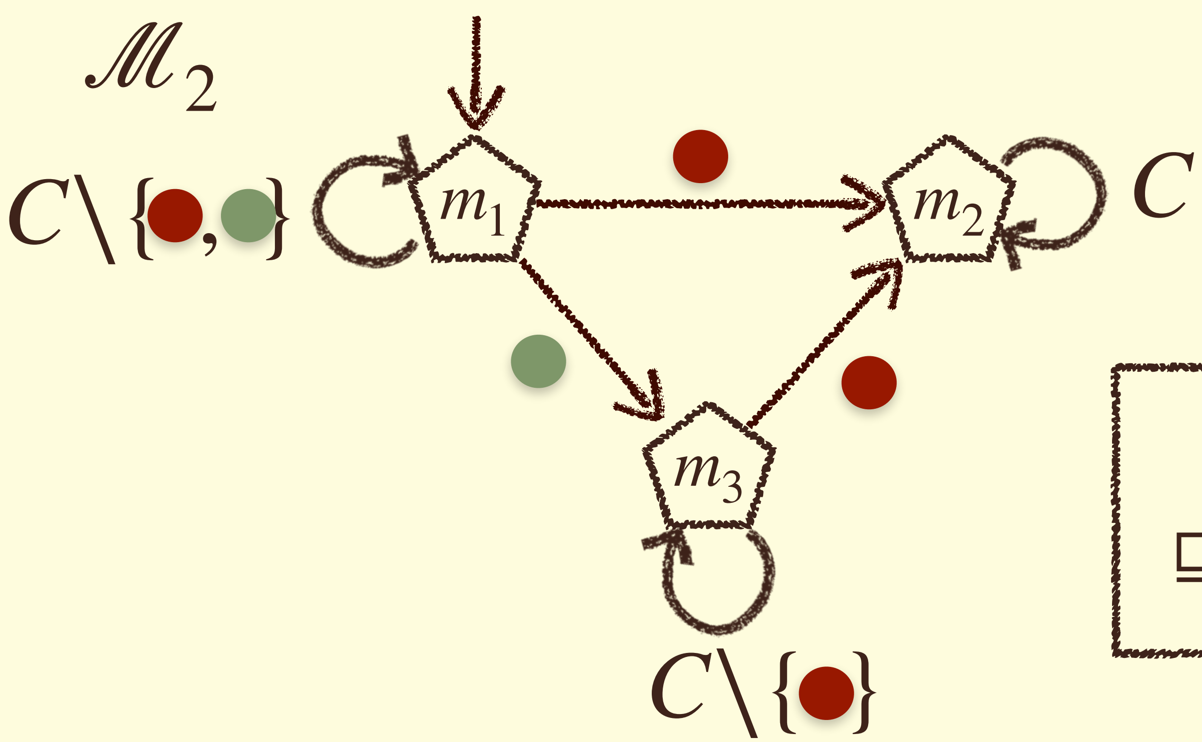

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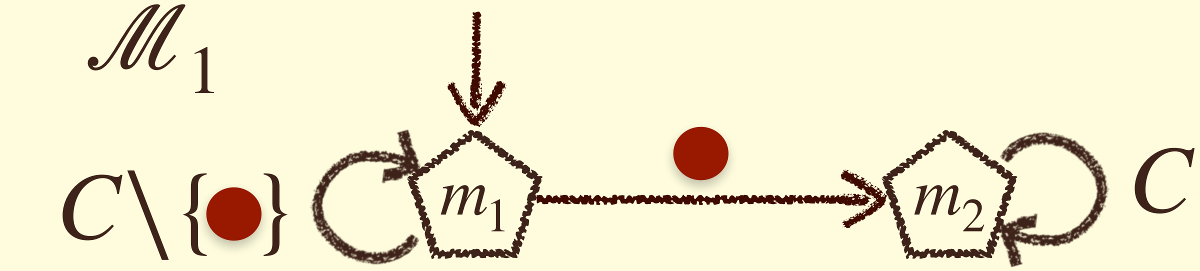

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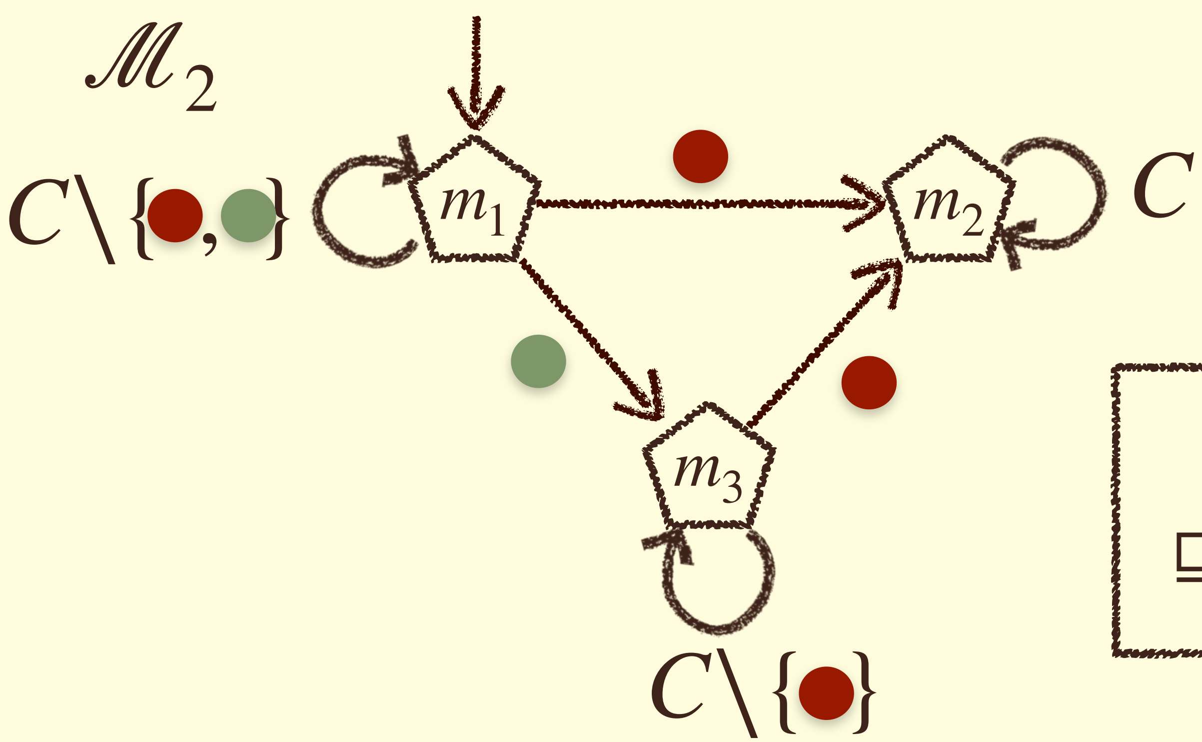
$\mathscr{M}_2$



$\sqsubseteq$ is $\mathscr{M}_2$-selective

$\sqsubseteq$ is $\mathscr{M}_1$-monotone and $\mathscr{M}_2$-selective
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➡ Memory $\mathscr{M}_2$ is sufficient for both players!!

Conclusion

A generalization of [GZ05]

- To arena-independent finite memory
- Applies to generalized reachability or parity, lower- and upper-bounded (multi-dimension) energy games

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Limitations

- Does only capture arena-independent finite memory
- Hard to generalize (remember counter-example)
- Does not apply to multi-dim. MP, MP+parity, energy+MP (infinite memory)

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Other approaches

- Sufficient conditions giving half-memory management results
- Compositionality w.r.t. objectives [LPR18]

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Further work

- Understand the arena-dependent framework
- Infinite arenas
- Probabilistic setting
- Other concepts (Nash equilibria)