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On Effective Representations of Well Quasi-Orderings

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Contents

Chapter 1

Introduction

Well Quasi-Ordering (WQO) is a notion from order theory that lies between well-orders (ordinals) and well-founded quasi-orderings: a well-order is WQO, and a WQO is well-founded. It enjoys several equivalent definitions (see Section 2.3), and has therefore been introduced several times independently, as people were interested in one of the characterization. Higman [1] studied orderings having the *finite basis property*: every subset has a non-empty but finite set of minimal elements (see (WQO4) in Section 2.3), as a generalization of well-orderings (every subset has exactly one minimal element). In the 1940's, Vazsonyi and Erdös were interested in conjectures of the form of (WQO1), and thus naturally introduced WQOs with this definition. The concept of WQOs have also been introduced as a strengthening of well-founded quasi-orderings: for instance, the powerset of a well-founded order (ordered with the majoring ordering, see Section 7.3) may not be well-founded. But it is if the QO one started with is WQO (see (WQO7) in Section 2.3). More generally, WQOs enjoy many more closure properties than well-founded orderings, letting them be easier to work with. This is I think the reason of the success of this notion in many areas of mathematics and computer science: Combinatorics, Topology, Automata Theory and Formal Languages, Proof Theory, Term Rewriting, Graph Theory, Program Verification, and more. See [2] for an early history of the concept and [3] for a recent survey on the use of WQOs in many areas of mathematics and computer science, written as a report of a Dagstuhl seminar bringing together researchers from many different communities of mathematics and computer science around the central notion of WQOs. Note that even though WQOs enjoy closure properties under many natural operations, a *better* notion has been introduced by Nash-Williams in the late 1960's, which enjoys even more closure properties. This notion, *Better Quasi-Orderings* is defined in Section 9.2.

Although WQO have proved their strength in many areas of computer science, our motivations will mainly come from Program Verification. Nonetheless, the results of this thesis are general and may be of interest for any computer scientist working with WQOs.

WQOs in Program Verification The efficiency of well-founded orderings to prove program termination, already suggested by Turing [4], is widely known. It is thus not a surprise that WQOs may be used to show program termination. More generally, several alternative definition of WQOs (see Section 2.3) contain a flavor of finiteness, which is what makes WQOs powerful object in computer science.

In the 1990's, Finkel, Schnoebelen, Abdullah and Jonsson introduced a large class of infinite state systems, *Well Structured Transition Systems (WSTS)*, which can be verified using generic methods conceptually similar to the ones used to verify finite state systems (see [5, 6] for surveys). The key ingredient in the notion of WSTS is the presence of a WQO compatible with the structure of the transition system. The finite basis property (*cf.* above) ensures that some infinite sets of states enjoy finite representations, and under mild effectiveness assumptions, we may perform computations with those sets, ultimately proving verification properties of the system (coverability, termination, boundedness, ...). Last year, the main contributors to this generic framework which has flourished during the last 20 years, have received the CAV award.

Outline. In the first part of this thesis, we define a notion of effectiveness for WQOs, and proceed to prove that a large class of WQOs is effective in this sense. Our notion of effectiveness includes most of the requirements needed for generic algorithms. In the second part, we study some logical aspect of WQOs.

Chapter 2

Basics of Order Theory and Well Quasi-Orderings

These chapter is devoted to defining the terminology we use throughout this manuscript. We also recall well-known facts from order-theory, along with the main ideas for their proofs, which are voluntarily kept concise. For a deeper presentation of these notions, we invite the reader to refer to [7, 2] for instance.

2.1 Preliminaries

Natural Numbers. The set of all natural numbers is denoted $\mathbb{N}$. When used on natural numbers, $\leq$ denotes the usual ordering on $\mathbb{N}$. The finite set $\{1, \ldots, n\}$ will be denoted $[n]$. A *permutation* is a bijection from $[n]$ to $[n]$. The set of all permutations $n!$, denoted S_n, is a group, sometimes called the *symmetric group*. It has $n!$ elements, which is asymptotically exponential in n (by Stirling formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$).

Sets. We use standard set-theoretic operations on sets: membership $x \in S$, inclusion $S \subseteq T$, union $S \cup T$, intersection $S \cap T$, set difference $S \smallsetminus T$. Given a set X, the set of all subsets of X is denoted $\mathcal{P}(X)$. When $S \in \mathcal{P}(X)$, the set difference $X \smallsetminus S$ is called the complement of S and will be denoted $\complement S$. If $\boldsymbol{A}$ is a set of sets, $\bigcup \boldsymbol{A}$ denotes the set $\bigcup_{S \in \boldsymbol{A}} S$ (a notation coming from set-theory).

Relations. A *relation* R on a set X is a subset of $X \times X$. As such, we use set-theoretic operations on relations, *e.g.* $R \subseteq S$ means $\forall x, y \in X.\ xRy \Rightarrow xSy$, where xRy denotes membership $(x, y) \in R$. Given a relation R on a set X and a function $f : X \to Y$, we also refer to the image of R by f to denote the relation S such that $f(x)Sf(y)$ if and only if $x, y \in X$ such that xRy.

We sometimes use the functional point of view, for instance the composition of two relations R_1 and R_2 is the relation $R = R_1 \circ R_2$ defined by $xRy \overset{\text{def}}{\Leftrightarrow} \exists z.\ xR_1z \wedge zR_2y$.

A relation is:

- *reflexive* if $\forall x \in X.\, xRx$,
- *transitive* if $\forall x, y, z \in X.\, xRy \wedge yRz \Rightarrow xRz$,
- *symmetric* if $\forall x, y \in X.\, xRy \Rightarrow yRx$,
- *antisymmetric* if $\forall x, y \in X.\, (xRy \wedge yRx) \Rightarrow x = y$.

An *equivalence relation* is a reflexive, transitive and symmetric relation. Let E be such a relation on X. Elements x and y in X such that xEy are called *equivalent*. For every $x \in X$, $[x]_E$, or $[x]$ when E is understood, denotes the *equivalence class* of X: $[x] = \{y \in X \mid xEy\}$. We say that x is a representant of $[x]$. Observe that $[x] = [y] \iff xEy$. As a consequence, two equivalence classes are either equal or disjoint. It follows that equivalence classes form a partition of X (conversely, any partition of X defines an equivalence relation).

The *quotient of* X *by* E, denoted X/E, is the set of equivalence classes of E.

A relation which is reflexive and transitive is called a *quasi-ordering*.

We will often use the abbreviations xRy, z and x, yRz to express $xRy \wedge xRz$ and $xRz \wedge yRz$, respectively.

Sequences. We assume some familarity with ordinals. Given an ordinal α, we denote by $\boldsymbol{\alpha}$ the set of strictly smaller ordinals, *i.e.* $\boldsymbol{\alpha} \stackrel{\text{def}}{=} \{\beta \mid \beta < \alpha\}$. A *sequence* over X is a function $\boldsymbol{s} : \boldsymbol{\alpha} \to X$ for some ordinal α. The *length* of a sequence $\boldsymbol{s}$, denoted $|\boldsymbol{s}|$, is the ordinal α, if $\boldsymbol{\alpha}$ is its domain. The only sequence of length 0, denoted ϵ, is also called the *empty sequence*. The set (or class) of sequences of length α is denoted X^α, and $X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$. The *concatenation* of two sequences $\boldsymbol{s}$ and $\boldsymbol{t}$ over the same set X of length α and β respectively is the sequence of length $(\alpha + \beta)$ denoted $\boldsymbol{s} \cdot \boldsymbol{t}$ and defined by $(\boldsymbol{s} \cdot \boldsymbol{t})(\gamma) = \boldsymbol{s}(\gamma)$ if $\gamma < \alpha$, and $(\boldsymbol{s} \cdot \boldsymbol{t})(\alpha + \gamma) = \boldsymbol{t}(\gamma)$ if $\gamma < \beta$. This operation is associative.

When $(X, \leq)$ is a quasi-ordered set (see below), sequences are quasi-ordered with the *embedding quasi-ordering*: if $\boldsymbol{s}$ and $\boldsymbol{t}$ are sequences of length α and β respectively, then

$$\boldsymbol{s} \leq \boldsymbol{t} \stackrel{\text{def}}{\Leftrightarrow} \exists f : \boldsymbol{\alpha} \to \boldsymbol{\beta} \text{ strictly increasing} .\, \forall \gamma < \alpha.\, \boldsymbol{s}(\gamma) \leq \boldsymbol{t}(f(\gamma))$$

When $\boldsymbol{s} \leq \boldsymbol{t}$, the function f is called a *witness* of the embedding of $\boldsymbol{s}$ in $\boldsymbol{t}$. A *subsequence* of a sequence $\boldsymbol{t}$ is a sequence $\boldsymbol{s}$ that embeds into $\boldsymbol{t}$ when the order considered on X is the equality.

A sequence is finite if its length is a natural number. The set of finite sequences over X is denoted X^* instead of $X^{<\omega}$, and the embedding quasi-ordering between finite sequences will be denoted $\leq_*$. Finite sequences are often described by the ordered list of their images: $\boldsymbol{u} = x_1 x_2 \cdots x_n \cdots$ where $x_i = \boldsymbol{u}(i-1)$. When X is a finite alphabet (*i.e.* a finite set ordered with equality), finite sequences over X are often called *finite words*. In Section 11.1, we freely use regular expressions like $(ab)^* + (ba)^*$ to denote regular languages. Given a letter $a \in X$, $|u|_a$ denotes the number of occurrences of a in u. A word $\boldsymbol{v}$ is a *factor* of $\boldsymbol{u}$ if there exist words $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ such that $\boldsymbol{u} = \boldsymbol{u}_1 \boldsymbol{v} \boldsymbol{u}_2$. If furthermore $\boldsymbol{u}_1 = \epsilon$ then $\boldsymbol{v}$ is a *prefix* of u, while if $\boldsymbol{u}_2 = \epsilon$ then $\boldsymbol{v}$ is a *suffix* of $\boldsymbol{u}$.

2.2 Order Theory

Orderings. Let X be a set. A *quasi-ordering* (abbreviated QO) on X is a reflexive and transitive relation, often denoted $\leq$, on X. If $x \leq y$, we say that x is smaller than y, or y is greater than x, which is also denoted $y \geq x$. If $\leq$ is also anti-symmetric, then it is called a *partial-ordering*, or often simply an *ordering*. Every quasi-order $(X, \leq)$ defines a partial-ordering on $X/\equiv$, where $\equiv$ is the equivalence relation defined by $\equiv = \leq \cap \geq$.

A quasi-ordering $(X, \leq)$ is said to be *total*, or *linear*, if every pair of elements is comparable: $\leq \cup \geq = X^2$. If $(X, \leq)$ is not linear, there are some incomparable elements. We define for this cases the relation $\perp$: $x \perp y \overset{\text{def}}{\Leftrightarrow} x \not\leq y \wedge y \not\leq x$. A *chain* Y of $(X, \leq)$ is a subset of X which is totally ordered, that is $(Y, \leq_Y)$ where $\leq_Y$ is the restriction of $\leq$ to Y, is a total (partial-) order. An *antichain* of X is a subset $S \subseteq X$ such that elements of S are pairwise incomparable: $\forall x \neq y \in S.\ x \perp y$. A QO $(X, \leq)$ is FAC (for *finite antichain condition*) if it has no infinite antichains.

Given a QO $(X, \leq)$, we define its associated *strict ordering*, denoted $<$, by $< = \leq \setminus \equiv$. The QO is *well-founded* if there are no infinite strictly decreasing sequences $x_1 > x_2 > x_3 > \ldots$ in $(X, \leq)$. If it is also antisymmetric and linear, then we say it is a *well-order*. Note that ordinals are exactly equivalence classes of well-orderings for order-isomorphism equivalence relation (*cf.* next paragraph).

An *extension* of a quasi-ordering $\leq$ on X is a quasi-ordering $\leq'$ also on X such that $\leq \subseteq \leq'$.

Mappings between Quasi-Ordered Sets. A mapping $f : X \to Y$ between two quasi-ordered sets $(X, \leq_X)$, $(Y, \leq_Y)$ is:

- *monotone* if for every $x, y \in X$, $x \leq_X y \Rightarrow f(x) \leq_Y f(y)$.
- a *reflection* if for every $x, y \in X$, $f(x) \leq_Y f(y) \Rightarrow x \leq_X y$
- an *embedding* of X into Y if it is a monotone reflection, that is for every $x, y \in X$, $x \leq_X y \iff f(x) \leq_Y f(y)$.
- An *isomorphism* if it is a bijective embedding.

As long as we are only interested in the properties of the quasi-ordering, two isomorphic QO are identical.

Closed Subsets. Given a subset $S \subseteq X$ of a QO $(X, \leq)$, we define:

- its *downward-closure* $\downarrow S \overset{\text{def}}{=} \{x \in X \mid \exists y \in S.\ x \leq y\}$
- its *upward-closure* $\uparrow S \overset{\text{def}}{=} \{y \in X \mid \exists x \in S.\ x \leq y\}$
- its *strict-downward-closure* $\downarrow_< S \overset{\text{def}}{=} \{x \in X \mid \exists y \in S.\ x < y\}$
- its *strict-upward-closure* $\uparrow_> S \overset{\text{def}}{=} \{y \in X \mid \exists x \in S.\ x < y\}$

When $(X, \leq)$ is not clear from the context, we use the more explicit notation $\uparrow_X S$ (*resp.* $\downarrow_X S$), or $\uparrow_\leq S$. Sometimes, we only subscript by a symbol that clearly refers to some specific QO, *e.g.* if the QO is $\leq_{\mathrm{st}}$, we write $\downarrow_{\mathrm{st}}$ instead of $\downarrow_{\leq_{\mathrm{st}}}$.

In the case S is a singleton, we write $\uparrow x$ for $\uparrow\{x\}$ and $\downarrow x$ for $\downarrow\{x\}$, when it causes no confusion (*e.g.* not when x is itself a set). With this abbreviation, $\uparrow S = \bigcup_{x \in S} \uparrow x$ and $\downarrow S = \bigcup_{x \in S} \downarrow x$. These relations are particularly interesting when S is a finite set.

A subset U of X is said to be *upward-closed* when $U = \uparrow U$. A subset D of X is said to be *downward-closed* when $U = \downarrow U$. We denote the set of all upward-closed sets of X by $Up(X) \subseteq \mathcal{P}(X)$ and the set of all downward-closed sets of X by $Down(X) \subseteq \mathcal{P}(X)$. Note that $Up(X)$ and $Down(X)$ are closed under union and intersection. Besides, the complement function from $\mathcal{P}(X)$ to itself is an isomorphism between $(Up(X), \supseteq)$ and $(Down(X), \subseteq)$, and it is an involution.

Irreducible Subsets. A subset $S \subseteq X$ is said to be *(up)-directed* if for every $x, y \in S$, there exists $z \in S$ such that $x \leq z$ and $y \leq z$. Similarly, S is said to be *(down)-directed* if for every $x, y \in S$, there exists $z \in S$ such that $x \geq z$ and $y \geq z$. When the adjective *directed* is used alone, it will always mean (up)-directed. A *filter* of X is a non-empty upward-closed (down)-directed subset of X. The set of filters of X is denoted $Filters(X)$. An *ideal* of X is a non-empty downward-closed (up)-directed subset of X. The set of ideals of X is denoted $Idl(X)$. Filters *are not* complements of ideals, and vice versa. Observe that for every $x \in X$, $\uparrow x \in Filters(X)$ and $\downarrow x \in Idl(X)$. Such subsets are respectively called *principal filters* and *principal ideals*.

The main property of filters and ideals is their irreducibility in $Up(X)$ and $Down(X)$ respectively. Formally, an upward-closed set $U \in Up(X)$ is *irreducible* if it is non-empty, and for every $U_1, U_2 \in Up(X)$, $U \subseteq U_1 \cup U_2$ implies that $U \subseteq U_1$ or $U \subseteq U_2$. Similarly, a downward-closed set D is *irreducible* if it is non-empty, and for every $D_1, D_2 \in Down(X)$, $D \subseteq D_1 \cup D_2$ implies $D = D_1$ or $D = D_2$. Irreducible sets are those that cannot be written as a finite union of the others: $D \in Down(X)$ is irreducible if and only if for every $D_1, \ldots, D_n \in Down(X)$, $D = D_1 \cup \cdots \cup D_n$ implies $D = D_i$ for some i. The statement also holds for irreducible sets of $Up(X)$.

The following proposition relates irreducibility and directedness:

Proposition 2.2.1. *Filters are exactly the irreducible sets of $Up(X)$ and ideals are exactly the irreducible sets of $Down(X)$.*

Proof. We provide a proof that a downward-closed set of X is an ideal if and only if it is irreducible. The proof for upward-closed sets is dual.

($\Rightarrow$) Let I be an ideal of X and $D_1, D_2 \in Down(X)$ such that $I \subseteq D_1 \cup D_2$. We show that if $I \not\subseteq D_1$, then $I \subseteq D_2$. Let $x \in I \setminus D_1 \subseteq D_2 \setminus D_1$. For any $y \in I$, there exists $z \in I$ such that $z \geq x, y$. Since D_1 is downward-closed, $z \notin D_1$. Therefore $z \in D_2$, and thus $y \in D_2$, since D_2 is downward-closed.

($\Leftarrow$) By contraposition, let $D \in Down(X)$ which is not an ideal, that is there exists $x, y \in D$ such that $\uparrow x \cap \uparrow y \cap D = \emptyset$, or equivalently $D \subseteq \complement(\uparrow x \cap \uparrow y)$. Define $D_1 = \complement \uparrow x$ and $D_2 = \complement \uparrow y$. Then $D_1 \cup D_2 = \complement(\uparrow x \cap \uparrow y)$ and thus $D \subseteq D_1 \cup D_2$. Besides, x and y are incomparable, since otherwise $\max(x, y)$ would belong to $D \cap \uparrow x \cap \uparrow y$. Therefore, $x \in D \setminus D_2$ and $y \in D \setminus D_1$. $\square$

2.3 Well Quasi Orderings

A QO $(X, \leq)$ is a *well-quasi ordering* (WQO for short) if and only if one of the following equivalent statement holds:

(WQO1)	For every infinite sequence $x_0, x_1, x_2, \ldots$ has an *increasing pair*, that is a pair $x_i \leq x_j$ for $i < j$. A sequence will be called *good* if it has an increasing pair, and *bad* otherwise. Thus in a WQO, all bad sequences are finite.
(WQO2)	Every infinite sequence $x_0, x_1, x_2, \ldots$ has an infinite increasing subsequence: $\exists i_0 < i_1 < \ldots$ such that $x_{i_0} \leq x_{i_1} \leq \ldots$
(WQO3)	$(X, \leq)$ is FAC and well-founded.
(WQO4)	Every non-empty subset $S \subseteq X$ has a non-empty and finite (up to equivalence) set of minimal elements. We denote by $\min(S)$ an arbitrarily chosen set of minimal elements of S.
(WQO5)	Every upward-closed set $U \in Up(X)$ is a finite union of principal filters.
(WQO6)	$(Up(X), \supseteq)$ is well-founded. This property is sometimes called the *Ascending Chain condition.*
(WQO7)	$(Down(X), \subseteq)$ is well-founded.

Proof. Of equivalence

(WQO2) $\Rightarrow$ (WQO1): trivial.

(WQO1) $\Rightarrow$ (WQO3): An antichain is a bad sequence, hence cannot be infinite. Similarly, a strictly decreasing sequence is a bad sequence, hence cannot be infinite.

(WQO3) $\Rightarrow$ (WQO2): This is the difficult implication. It relies on the Infinite Ramsey Theorem. Let $(x_n)_{n\in\mathbb{N}}$ be an infinite sequence of elements of X. We color the infinite complete graph $G = (\mathbb{N}, \{\{i, j\} \mid i \neq j \in \mathbb{N}\})$ with the three colors $\{\leq, >, \perp\}$. As expected, a two-element subset $\{i, j\}$ with $i < j \in \mathbb{N}$ is colored with $\leq$ if $x_i \leq x_j$, with $>$ if $x_i > x_j$, and with $\perp$ otherwise. Now, the Infinite Ramsey Theorem states that G has an infinite monochromatic clique. An infinite clique of G colored with $>$ induces a strictly decreasing infinite subsequence of $(x_n)_{n\in\mathbb{N}}$, which is impossible since $(X, \leq)$ is well-founded. An infinite clique of G colored with $\perp$ induces an infinite antichain in X, which is also impossible. Therefore, there is in G an infinite clique colored with $\leq$, *i.e.* an infinite increasing subsequence of $(x_n)_{n\in\mathbb{N}}$.

(WQO3) $\Leftarrow$ (WQO4): Let S be a subset of X. Since $(X, \leq)$ is well-founded, S has a non-empty subset of minimal elements. Furthermore, non-equivalent minimal elements form an antichain, thus S has a non-empty and finite (up to equivalence) number of minimal elements.

(WQO3) $\Leftarrow$ (WQO4): Since any subset of X has a non-empty subset of minimal elements, $(X, \leq)$ is well-founded. Besides, for any antichain A, $A = \min(A)$, and thus A is finite.

(WQO4) $\Rightarrow$ (WQO5): An upward-closed set U is in particular a subset of X. Thus $\min(U)$ is finite and non empty. By definition of minimal, $U = \uparrow\min(U) = \bigcup_{x\in\min(U)} \uparrow x$.

(WQO4) $\Leftarrow$ (WQO5): Conversely, for an arbitrary subset $S \subseteq X$, the decomposition of the upward-closed set $\uparrow S = \bigcup_{i=1}^{n} \uparrow x_i$ as a finite union of filters gives the minimal elements of S.

(WQO6) ⇔ (WQO7): follows from the prior observation on $\complement$ being an isomorphism between $(Up(X), \supseteq)$ and $(Down(X), \subseteq)$ (*cf.* end of paragraph on Closed Subsets).

(WQO2) ⇒ (WQO6): given a strictly increasing sequence $U_0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots$ of upward-closed sets, define a sequence of elements of X as follows: $x_0 \in U_0$, and for all i, $x_i \in U_i \setminus U_{i-1}$. Such a sequence exists since (U_n) is strictly increasing. Moreover, the sequence (x_n) is bad, hence finite. Indeed, let $i < j$, $x_j \in U_j \setminus U_{j-1}$, and $U_i \subseteq U_{j-1}$, thus $u_j \notin U_i$. But U_i is upward-closed, so $x_i \leq x_j$ would imply $x_j \in U_i$.

(WQO7) ⇒ (WQO1): given an infinite sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X, we define a decreasing sequence of downward-closed sets $D_i = \bigcap_{j\geq i} \downarrow x_j$. The sequence $(D_n)_{n\in\mathbb{N}}$ is decreasing for $\subseteq$ by construction. Thus, there exists $i \in \mathbb{N}$ such that $D_i = D_{i+1}$. In particular, for every $\downarrow x_i \subseteq \downarrow x_j$ for some $j > i$, which is equivalent to $x_i \leq x_j$. □

The property that bad sequences are finite is of course very useful in computer science, for instance in the context of proving termination of programs. However, such a finiteness property already lies in the definition of well-foundedness. What makes WQO a more practical tool than well-founded orderings is that the notion is preserved under many operations on quasi-ordered sets. Cartesian products (with componentwise quasi-ordering [8]) and finite sequences (with the Higman or subword quasi-ordering [1]) are the most prominent examples. In Part I, many such constructions will be presented in details. In addition to these, we would like to mention that several quasi-orderings on trees labeled with elements of a WQO are WQOs themselves. And of course, the famous Robertson-Seymour Theorem states that the minor ordering on graphs (labeled by elements in a WQO) is a WQO [9].

Properties of Closed Subsets of a WQO. The following theorem is due to Bonnet and proved in [10]. Similar proofs can also be found in [11, 12]. To the best of the author's knowledge, the proof presented below is new.

Theorem 2.3.1 (Bonnet)**.** *A QO $(X, \leq)$ is FAC if and only if every downward-closed set of X is a finite union of ideals, if and only if every upward-closed set of X is a finite union of filters.*

Proof. A QO $(X, \leq)$ is FAC if and only if $(X, \geq)$ is FAC. Upward-closed sets of $(X, \leq)$ being downward-closed sets of $(X, \geq)$, and filters of $(X, \leq)$ being ideals of $(X, \leq)$, we only need to show the first equivalence.

(⇐) Let $S \subseteq X$ be an infinite subset of X. If the downward-closure $\downarrow S$ has an ideal decomposition $\downarrow S = I_1 \cup \cdots \cup I_n$, then there must be infinitely many $x \in S$ that belong to the same ideal I_{i_0}. Take x, y two elements of S that belong to I_{i_0}, by directedness there exists $z \in I_{i_0}$ such that $z \geq x, y$. But $I_{i_0} \subseteq \downarrow S$, therefore there exists $t \in S$ such that $t \geq z \geq x, y$, and S is not an antichain.

(⇒) By contraposition, let $D \in Down(X)$ that does not admit a finite ideal decomposition. We apply Zorn's Lemma to the quasi-ordered set $(\mathcal{P}(\mathcal{P}(D) \cap Idl(X)), \leq_c)$, where $\leq_c$ denotes the *covering quasi-ordering* defined by: $\boldsymbol{A} \leq_c \boldsymbol{B}$ iff $\bigcup \boldsymbol{A} \subseteq \bigcup \boldsymbol{B}$ iff $\forall I \in \boldsymbol{A}.\ I \subseteq \bigcup \boldsymbol{B}$. Note that $\mathcal{P}(D) \cap Idl(X) = Idl(D, \leq)$, where $(D, \leq)$ is the QO obtained by restricting $(X, \leq)$ to D (this relies on the fact that D is downward-closed, the structure of the ideals of a restriction is more complex in general, see Section 4.3). Therefore, given $\boldsymbol{A}$ a set of ideals of D, $\bigcup \boldsymbol{A}$ is a downward-closed subset of D, and $\leq_c$ corresponds to inclusion for downward-closed sets.

In order to apply Zorn's Lemma, we show that every chain of $\mathcal{P}(Idl(D))$ has an upper bound. Given such a chain $\boldsymbol{C}$, define $\boldsymbol{A} = \bigcup \boldsymbol{C} \subseteq Idl(D)$. For $\boldsymbol{B} \in \boldsymbol{C}$, $\boldsymbol{B} \leq_c \boldsymbol{A}$ since $\boldsymbol{B} \subseteq \boldsymbol{A}$. Therefore, by Zorn's Lemma, $(\mathcal{P}(\mathcal{P}(D) \cap Idl(X)), \leq_c)$ has a maximal element $\boldsymbol{M} \subseteq Idl(D)$, that is for every $\boldsymbol{A} \subseteq Idl(D)$, $\boldsymbol{M} \not<_c \boldsymbol{A}$. In particular, for any $I \in \boldsymbol{M}$, $\boldsymbol{M} \not<_c (\boldsymbol{M} \smallsetminus \{I\})$, *i.e.* $\exists K \in \boldsymbol{M}.\ K \not\subseteq \bigcup_{J \in \boldsymbol{M} \smallsetminus \{I\}} J$. Trivially, $K \subseteq \bigcup_{J \in \boldsymbol{M} \smallsetminus \{I\}} J$ for $K \in \boldsymbol{M} \smallsetminus \{I\}$. Therefore, for the above condition to be true, the following must hold: $I \not\subseteq \bigcup_{J \in \boldsymbol{M} \smallsetminus \{I\}} J$. In other words, $\forall I \in \boldsymbol{M}.\ \exists x_I \in I.\ \forall J \in \boldsymbol{M}.\ (J \neq I \Rightarrow x_I \notin J)$. Using the axiom of choice, we can chose such an element x_I for every $I \in \boldsymbol{M}$. The resulting set $\{x_I \mid I \in \boldsymbol{M}\}$ is obviously an antichain of $(X, \leq)$. It remains to show that $\boldsymbol{M}$ is infinite. For this, we show that $\bigcup \boldsymbol{M} = D$, and since we assumed that D cannot be decomposed as a finite union of ideals, it follows that $\boldsymbol{M}$ is infinite. For the sake of contradiction, assume $\bigcup \boldsymbol{M}$ is a strict subset of D. Then let $x \in D \smallsetminus \bigcup \boldsymbol{M}$, we have $\boldsymbol{M} <_c \boldsymbol{M} \cup \{\downarrow x\}$, contradicting the maximality of $\boldsymbol{M}$. □

Since a WQO is in particular a FAC QO, it follows that downward-closed and upward-closed sets of a WQO are finite union of ideals and filters, respectively. Actually, there is a more direct and much easier proof of this fact in the case of a WQO. For upward-closed sets, the decomposition already follows from (WQO5) above. Note that this even proves that upward-closed sets are finite unions of principal filters, and indeed, all filters are principal in a WQO.

For downward-closed sets, the decomposition can be proved by induction on $(Down(X), \subseteq)$, which is well-founded when $(X, \leq)$ is a WQO (*cf.* (WQO7)). Actually, the general case is often proved using this special case (*cf.* [10, 11, 12]).

Note that the situation is not symmetric between upward-closed and downward-closed sets of a WQO: unlike for filters, not all ideals are principal. Indeed, (WQO5) shows that all filters of $(X, \leq)$ are principal when $(X, \leq)$ is a WQO. Therefore, all ideals are principal if and only if $(X, \geq)$ is a WQO. But $\leq$ and $\geq$ are simultaneously WQOs if and only if X is finite.

Even though not all ideals are principal, principal ideals convey the good intuition: an ideal is always the downward-closure $\downarrow x$ of some element x, however sometimes this element x is not in X: it is a limit point. When X is countable, this intuition can be made formal: ideals are exactly the downward-closure of chains. That is, ideals are limits of increasing sequences. More generally, a partially-ordered set $(X, \leq)$ can always be seen as a topological set $(X_a, \mathcal{O})$ equipped with the Alexandroff topology. In this setting, a topological operation called *sobrification* that essentially consists in adding the "missing limits" applied to $(X_a, \mathcal{O})$ corresponds exactly to the set of ideals. When $(X, \leq)$ is WQO, $(X_a, \mathcal{O})$ is Noetherian (for the Alexandroff topology). But topological spaces can be Noetherian for other topologies, in which case they may not correspond to some WQO. Noetherian spaces therefore constitute a generalization of WQOs, closed under more operations (*e.g.* infinite powerset). For more details, see [13] and references therein.

Cardinalities Let $(X, \leq)$ be a WQO. From the fact that all filters are principal, it follows that $(\mathit{Filters}(X), \subseteq)$ is order-isomorphic to $(X/\equiv, \leq)$, where $\equiv$ is the equivalence relation induced by $\leq$. Besides, the irreducibility of filters (Proposition 2.2.1) entails that $(\mathit{Up}(X), \subseteq)$ embeds in $(\mathcal{P}_f(\mathit{Filters}(X)), \sqsubseteq_{\mathcal{H}})$, where $\mathcal{P}_f$ denotes the finitary powerset and $\sqsubseteq_{\mathcal{H}}$ denotes the *Hoare* quasi-ordering defined by: For $\boldsymbol{A}, \boldsymbol{B} \subseteq \mathcal{P}_f(\mathit{Filters}(X))$, $\boldsymbol{A} \sqsubseteq_{\mathcal{H}} \boldsymbol{B} \overset{\text{def}}{\Leftrightarrow} \forall F \in \boldsymbol{A}.\ \exists F' \in \boldsymbol{B}.\ F \subseteq F'$. Furthermore, $(\mathit{Up}(X), \subseteq)$ is isomorphic to the quotient $\mathcal{P}_f(\mathit{Filters}(X))/\equiv_{\mathcal{H}}$, where $\equiv_{\sqsubseteq_{\mathcal{H}}} = \mathcal{H} \cap \sqsupseteq_{\mathcal{H}}$. More details on the Hoare quasi-ordering are given in Section 7.3.

From this isomorphism, it follows that if $X/\equiv$ is infinite, $\mathit{Up}(X)$ has the same cardinality as X. Since $\mathit{Down}(X)$ is isomorphic to $\mathit{Up}(X)$, it also has the same cardinality. Finally, $(\mathit{Down}(X), \subseteq)$ is isomorphic to $(\mathcal{P}_f(\mathit{Idl}(X))/\equiv_{\mathcal{H}}, \sqsubseteq_{\mathcal{H}})$ (*cf.* Proposition 2.2.1 as well), and thus $\mathit{Idl}(X)$ also have the same cardinality. If X is finite, all ideals are principal and thus $\mathit{Idl}(X)$ is isomorphic to $X/\equiv$.

Theorem 2.3.1 was originally proved to obtain these results on cardinalities [10]. In this manuscript, all the WQOs we consider happen to be countable.

Part I

The Ideal Approach to Computing with Closed Sets

Joint work with J.Goubault-Larrecq, P. Karandikar, N. Narayan Kumar and Ph. Schnoebelen

The general idea behind many generic algorithms from WSTS theory is a fixpoint computation of an infinite closed subset, *e.g.* the set of *coverable* configurations. Classically, these algorithms are described in term of upward-closed sets, that are representable by their finite basis (WQO5). More recently, many results suggested that it was more beneficial to work with downward-closed sets: the *coverability set* of a WSTS is downward-closed, and it is a central object in the theory [14, 15]; and it has been shown that computing with downward-closed sets provides better running-time in practice, notably using acceleration techniques [16, 17, 18, 19, 20, 21, 22, 23].

The issue is that downward-closed sets do not enjoy the same property as upward-closed set of having a finite number of maximal elements: for instance in $\mathbb{N}^2$ ordered with the product ordering $\langle n, m\rangle \leq_\times \langle n', m'\rangle \overset{\text{def}}{\Leftrightarrow} n \leq n' \wedge m \leq m'$, the subset $[0, 3] \times \mathbb{N}$ is downward-closed, but has no maximal elements. In the particular case of $\mathbb{N}^k$, it has first been observed in [16] that downward-closed sets could be represented finitely as downward-closure of elements in $(\mathbb{N} \cup \{\omega\})^k$. Later in [19], a similar property has been observed for the set of finite sequences over a finite alphabet, ordered with the embedding relation: downward-closed sets can be represented using *simple regular expressions*. These two positive examples lead to a general solution: the notion of *Adequate Domain of Limits* (see [24]). An adequate domain of limits essentially is a generalization of the two cases above: it is a set of limit elements missing from X to be able to express every downward-closed sets. Assuming the existence of such an adequate domain of limits, generic algorithms computing with downward-closed sets can be designed.

Ideals, a frequently rediscovered concept. Although it has long been known that downward-closed sets of a WQO can be decomposed as finite union of ideals (Theorem 2.3.1), the notion of ideals has only been brought to the domain of Verification quite recently, in [13]. In this article, it is proven that the set of ideals is an adequate domain of limits, and the smallest one, that is any adequate domain of limit embeds the set of ideals. Moreover, the extra elements introduced to handle downward-closed sets of $\mathbb{N}^k$ and A^* are exactly the ideals of these WQOs. This is not the first time the notion of ideals is rediscovered, [25] proves that downward-closed sets are union of ideals with a completely different terminology, following observations he had made in the case of finite graphs under the minor ordering.

Since their appearance in the domain of verification, ideals have several times proved themselves to be the right notion. Their structure carries more information than filters, which can be useful to analyze the complexity of algorithms relying on the well-foundedness of $Down(X)$ [26, 27]. Some important results with difficult and adhoc proofs have successfully been rethought in terms of ideals [28, 26], which leaves more options for generalizations. Besides, since ideals are intuitively limit elements of X, the structure of $Idl(X)$ is often close to that of X, this is essential in Section 9.4 for instance. This also offers the possibility to define the *completion* of a WSTS [29], whose states are ideals of the original states. Ideals were also applied to *separability of languages* [30, 31].

Contributions In WSTS theory, the notion of *effective WQO* has almost as many definitions as occurrences. At the very least, elements of an effective WQO can be computationally represented and compared (*i.e.* $\leq$ is decidable for the chosen representation of elements of X). However, no unified framework has been defined: authors usually simply gather every assumption they need for the design of their particular algorithm, and later proceed to show that these assumptions are trivially satisfied in their application. Our purpose here is to free people from this burden: we define a notion of effective WQO that should contain most of the ones present in the literature, and prove that a large class of WQOs, notably including most of the natural WQOs encountered in computer science, is effective. Our definition entails the computability of several operations on WQOs and their closed subsets which should be enough for the design of most of the algorithms that use WQOs.

More precisely, our definition assumes that we can represent elements of X, but also ideals of X, and that we can compare them. From there, we can represent upward-closed and downward-closed sets decomposing them in union of filters/ideals. We further assume the computability of all set-theoretic operations: union, intersection and complement.

As stated before, our motivations comes from verification, but the results are applicable anywhere.

Related Work This approach has been developed in [13], and our work is strongly inspired by theirs. On the one hand, our work is less general: we deal with WQOs and constructions preserving those, while they work with the more general notion of Noetherian spaces, which is preserved under constructions that do not preserve WQO. On the other hand, we believe our presentation is more suited for computer scientists: it deals more specifically with WQOs, hence requires less knowledge of advanced notions from topology, which also allows us to describe more precisely how to compute set-theoretic operations, and their complexity. That is, we hope our presentation is closer to an actual implementation, and provide relevant insight to this end.

At the end of some sections, we provide references for earlier related results. When no such references are given, it means that our results are novel, as far as we know.

Alternative Representations of Closed Subsets As noted before, assuming solely that elements of a WQO $(X, \leq)$ can be represented and the decidability of $\leq$, we can already represent and compare upward-closed sets, using their finite basis. It is then always possible to represent downward-closed sets by their complement. We call it the *excluded minor representation*: a downward-closed set D is represented by the minimal set of elements it does not contain. This may seem simpler than extra assumptions on the representability of ideals, but this representation has some drawbacks. First it breaks the symmetry between upward-closed and downward-closed sets. Moreover, union of downward-closed subsets then corresponds to intersection of upward-closed sets, which may be costly, while we would like union to be the most basic operation on closed subsets. Besides, we show in Section 8.3 that we may not be able to distinguish ideals among downward-closed sets with this representation. Sometimes, ideals carry valuable information that may not be directly read on their complement [26, 27].

Another issue is the size of the representations of closed subsets. For instance in Section 6.1, we prove that set-theoretic operations require exponential-time to be computed, by showing that the output of those operations might be exponential. This comes from our representation as union of filters/ideals that can become quite large. Many practical tools suffer from this explosion of the size of the representation of closed subsets, solutions to this problem are investigated in [32].

Chapter 3

Ideally Effective Well Quasi-Orders

3.1 Formal Definition

As mentioned in the previous chapter, our goal is to represent and compute with closed subsets of WQOs. In this chapter, we introduce ideally effective WQOs. The notion is defined as a list of axioms to be satisfied by a given WQO. We then argue how an ideally effective WQO in the sense of our definition satisfies the properties we are interested in, namely closed subsets can be handled algorithmically. The minimality of the definition, that is the redundancy of some axioms, is discussed in Chapter 8.

Definition 3.1.1 (Effective WQOs)**.** *A WQO* $(X, \leq)$ *is* effective *if it satisfies the following axioms:*

(XR) *There is a computational representation for* X. *In particular, membership in* X *is decidable.*

(OD) *The quasi-ordering* $\leq$ *is decidable (for the above representation).*

(IR) *There is a computational representation for the ideals of* X. *In particular, membership in* $Idl(X)$ *is decidable.*

(ID) *Inclusion of ideals is decidable.*

The datatypes and procedures witnessing these axioms will be called a presentation *of* $(X, \leq)$.

As can be inferred from the above definition, all the WQOs we consider are countable, and thus their sets of ideals are also countable (*cf.* end of Chapter 2). The definition restricts the range of representations for such countable X and $Idl(X)$ to representations satisfying (OD) and (ID).

Given an effective WQO, we fix the representation of closed subsets: upward-closed sets shall be represented as finite sets of elements of X, and downward-closed sets as finite sets of ideals. The semantics of such a representation is given by the union of the elements of the set: the syntax $S = \{x_1, \ldots, x_n\}$ for $x_i \in X$ denotes the upward-closed set $U = {\uparrow}S = \bigcup_{i=1}^{n} {\uparrow} x_i$, and the syntax $\boldsymbol{T} = \{I_1, \ldots, I_n\}$ for $I_i \in \mathit{Idl}(X)$ denotes the downward-closed set $D = \bigcup_{i=1}^{n} I_i$. Every closed set can be represented with this syntax, according to Theorem 2.3.1.

Inclusion between upward-closed sets (*resp.* downward-closed sets) thus reduces to a quadratic number of calls to the procedure given by (OD) (*resp.* (ID)): given $F_1, \ldots, F_n, F'_1, \ldots, F'_m$ filters of X, $\bigcup_{i=1}^{n} F_i \subseteq \bigcup_{j=1}^{m} F'_j$ if and only if $\forall i \in [n].\ \exists j \in [m].\ F_i \subseteq F'_j$. This relies on the fact that filters (*resp.* ideals) are the irreducible sets of $\mathit{Up}(X)$ (*resp.* $\mathit{Down}(X)$, Proposition 2.2.1).

Observe that this representation of closed subsets relies on the isomorphisms presented at the end of Chapter 2: $(\mathit{Up}(X), \subseteq)$ is isomorphic to $(\mathcal{P}_f(\mathit{Filters}(X))/\equiv_{\mathcal{H}}, \sqsubseteq_{\mathcal{H}})$ and $(\mathit{Down}(X), \subseteq)$ is isomorphic to $(\mathcal{P}_f(\mathit{Idl}(X))/\equiv_{\mathcal{H}}, \sqsubseteq_{\mathcal{H}})$. In all generality, the *Hoare quasi-ordering* is defined over subsets of a QO $(X, \leq)$ by:

$$S \sqsubseteq_{\mathcal{H}} T \overset{\text{def}}{\Leftrightarrow} \forall x \in S.\ \exists y \in T.\ x \leq y$$

for $S, T \subseteq X$. This quasi-ordering is not antisymmetric, even if $\leq$ is: for any $S, T \in \mathcal{P}(X)$, $S \sqsubseteq_{\mathcal{H}} T \iff {\downarrow}S \subseteq {\downarrow}T$. It follows from this characterization that for any $S \in \mathcal{P}_f(X)$, $S \equiv_{\mathcal{H}} {\downarrow}S \equiv_{\mathcal{H}} \max(S)$, where as usual $\equiv_{\mathcal{H}} = \sqsubseteq_{\mathcal{H}} \cap \sqsupseteq_{\mathcal{H}}$. On the other hand, $(\mathit{Up}(X), \subseteq)$ (*resp.* $(\mathit{Down}(X), \subseteq)$) is antisymmetric, and this is the reason it is not isomorphic to $(\mathcal{P}_f(\mathit{Filters}(X)), \sqsubseteq_{\mathcal{H}})$ (*resp.* $(\mathcal{P}_f(\mathit{Idl}(X)), \sqsubseteq_{\mathcal{H}})$) but to its associated partial-ordering.

In other terms, there are several syntaxes in $\mathcal{P}_f(\mathit{Idl}(X))$ for a given downward-closed set, and it is convenient to choose a canonical one.

Definition 3.1.2. *A representation* $\{I_1, \ldots, I_n\}$ *of a downward-closed set* D *is* canonical *if:*

$$\forall i, j \in [n].\ i \neq j \Rightarrow I_i \not\subseteq I_j$$

i.e. *downward-closed sets* D *are represented by the maximal elements of* $\{I \in \mathit{Idl}(X) \mid I \subseteq D\}$.

For upward-closed sets, we further use the isomorphism between $(\mathit{Filters}(X), \subseteq)$ and $(X, \geq)$. That is, $(\mathit{Up}(X), \subseteq)$ is isomorphic to $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{H}}{}^{\geq})$. Therefore, the maximal filters (for inclusion) are those represented by the minimal elements (for $\leq$).

Definition 3.1.3. *A representation* $\{x_1, \ldots, x_n\}$ *of an upward-closed set* U *is* canonical *if:*

$$\forall i, j \in [n].\ i \neq j \Rightarrow x_i \not\geq x_j$$

i.e. *upward-closed sets* U *are represented by* $\min(U)$.

Note that canonical representations are not unique: if $(X, \leq)$ is not antisymmetric, equivalent elements (for $\equiv = \leq \cap \geq$) can be used interchangeably. Moreover, the data-structure provided by (XR) and (IR) may be redundant (several representations for a single element). In all the concrete cases considered in the next chapters, we define a canonical syntax for every element of X and every ideal of $\mathit{Idl}(X)$ that are computable from any other representation of a given element or ideal.

Now that representations for closed sets are fixed, we can discuss the computability of set-theoretic operations on those sets. Note that union is trivial with our representation.

Definition 3.1.4 (Ideally effective WQOs)**.** *An effective WQO* $(X, \leq)$ *is* ideally effective *if its presentation satisfies the following:*

(IM) *Ideal membership is decidable: given* $x \in X$ *and* $I \in Idl(X)$*, does* $x \in I$ *?*

and the following functions are computable:

(PI) *Principal ideals:* $\begin{cases} X & \to & Idl(X) \\ x & \mapsto & \downarrow x \end{cases}$

(CF) *Complementing filters:* $\begin{cases} X & \to & Down(X) \\ x & \mapsto & \complement \uparrow x \end{cases}$

(CI) *Complementing Ideals:* $\begin{cases} Idl(X) & \to & Up(X) \\ I & \mapsto & \complement I \end{cases}$

(IF) *Intersecting Filters:* $\begin{cases} X \times X & \to & Up(X) \\ (x, y) & \mapsto & \uparrow x \cap \uparrow y \end{cases}$

(II) *Intersecting Ideals:* $\begin{cases} Idl(X) \times Idl(X) & \to & Down(X) \\ (I, J) & \mapsto & I \cap J \end{cases}$

(XF) *Filter Decomposition of* X*:* $\begin{cases} \emptyset & \to & Up(X) \\ () & \mapsto & X \end{cases}$

(XI) *Ideal Decomposition of* X*:* $\begin{cases} \emptyset & \to & Down(X) \\ () & \mapsto & X \end{cases}$

A presentation of $(X, \leq)$ *further equipped with the procedures witnessing the axioms above will be called a* full presentation *of* $(X, \leq)$*. A full presentation of an ideally effective WQO is said to be* polynomial-time *if all its procedures are running in polynomial-time.*

Note that given a WQO $(X, \leq)$, the two lasts axioms (XF) and (XI) are trivially satisfied, since constants are always computable. These axioms will find their importance in Section 3.3.

As in the case of inclusion, all set-theoretic operations on closed sets can be distributed over the unions, and therefore reduces to operations over filters and ideals. Therefore, if $(X, \leq)$ is ideally effective, all operations are computable, not only on filters and ideals, but more generally on closed sets. To this regard, (XF) and (XI) are important to complement empty closed sets (empty union of filter/ideals).

Note that axioms (IM) and (PI) have no counter-part for upward-closed sets since they are trivial then: $x \in \uparrow y \iff x \leq y$ and $\downarrow x$ is represented as $\{x\}$. In concrete examples, the structure of ideals is close to the one of X, and (IM) and (PI) will be trivial as well.

Although our definition of ideally effective WQOs does contain some redundancies (presented in Chapter 8), we rather work with the long version presented in this chapter because on concrete examples, we are not only interested in computability, but also efficiency. The only exception is axiom (IM), for which it is equivalent to compose (PI) and (ID) in all the cases considered in the subsequent chapters. More precisely, we rely on the equivalence $x \in I \iff \downarrow x \subseteq I$, for $x \in X, I \in Idl(X)$ and $(X, \leq)$ a WQO.

3.2 Basic Ideally Effective WQOs

3.2.1 Finite Quasi-Orderings

The simplest WQOs one can think of probably are *finite alphabets*. They consist of a finite set A ordered with equality. Most of the time it is used as a starting point to build more complex WQOs, as in the case of finite sequences with the Higman ordering (Section 6.1), extensively studied in language theory. It is also used in verification to order states of well-structured transition systems.

This WQO $(A, =)$ is ideally effective: one can for instance represent elements of A using natural numbers up to $|A| - 1$. The ordering (equality) is trivially decidable. Ideals of $(A, =)$ all are principal, that is of the form $\downarrow x$ for $x \in A$. We thus represent ideals as elements, same as we do for filters. Therefore, ideal inclusion coincides with the ordering, and (PI) is given by the identity function. All other operations are trivial: intersection of filters (*resp.* ideals) is always empty except if the two filters (*resp.* ideals) are equal, and $\complement \uparrow x = \complement \downarrow x = A \smallsetminus \{x\}$.

More generally, any finite WQO $(X, \leq)$ given in an adequate form (*e.g.* the quasi-ordering can be described as a matrix) is ideally effective. All ideals are principal, which is no surprise since $(X, \geq)$ is also a WQO, and its filters are the ideals of $(X, \leq)$. The four set-theoretic operations can be computed by brute force enumeration of the elements of X.

3.2.2 Natural Numbers

Natural numbers $(\mathbb{N}, \leq)$ is among the most frequently occurring WQOs in computer science. Observe that since $\leq$ is linear, any downward-closed set is actually an ideal, except the empty set $\emptyset$. There are two kinds of downward-closed sets in $\mathbb{N}$: those that are bounded, *i.e.* of the form $\downarrow n$, and the whole set $\mathbb{N}$ itself. The first kind constitutes all the principal ideals. The second kind is often denoted $\downarrow \omega$, for instance in [16]. Ideal inclusion is thus decidable: principal ideals are compared as the elements, and $\downarrow \omega$ is greater than all the others. Hence, ideals of $(\mathbb{N}, \leq)$ are linearly ordered, which makes intersections trivial: it consists of the maximum for filters and of the minimum for ideals. Finally, complements are computed as follows:

$$\complement \uparrow (n+1) = \downarrow n \qquad \complement \downarrow n = \uparrow (n+1)$$
$$\complement \uparrow 0 = \emptyset \qquad \complement \downarrow \omega = \emptyset$$

3.2.3 Ordinals

The prior analysis can be extended to any linear WQO, *i.e.* any ordinal. For the rest of this section, we assume basic knowledge of ordinals. Given an ordinal α, we write $\boldsymbol{\alpha}$ for the set of ordinals $\{\beta \mid \beta \leq \alpha\}$, in accordance with the classical set-theoretic construction of ordinals.

Let $(X, \leq) = (\boldsymbol{\alpha}, \leq)$. Once again, X being linearly ordered, its ideals are its downward-closed sets (except $\emptyset$). Therefore, there are three types of ideals:

1. $I = X$,
2. I has a maximal element $\beta \in X$, in which case $I = \downarrow \beta$,
3. Or I has a supremum $\beta \in X \smallsetminus I$, in which case $I = \downarrow_< \beta = \boldsymbol{\beta}$.

Note that in the second case, $I = \downarrow \beta = \downarrow_< (\beta + 1) = \boldsymbol{\beta + 1}$. Thus every ideal of $(X, \leq)$ is a $\boldsymbol{\beta}$ for some $\beta \in \boldsymbol{\alpha + 1} \smallsetminus 0$, and ideal inclusion coincides with the natural ordering on $\boldsymbol{\alpha + 1}$.

Now, assuming that we can represent elements of X in a way that makes $\leq$ decidable, $(X, \leq)$ is ideally effective. Indeed, a representation for $\boldsymbol{\alpha}$ and a decision procedure for $\leq$ are easily extended to $(\boldsymbol{\alpha + 1}, \leq)$. Therefore, ideal inclusion is decidable for X. Intersections are computable as the maximum for filters, minimum for ideals. Finally, complements are computed as follows:

$$
\begin{aligned}
&\complement \uparrow \beta = \boldsymbol{\beta} \qquad && \complement \boldsymbol{\beta} = \uparrow \beta \qquad && \text{for } \beta \in \boldsymbol{\alpha} \\
&\complement \uparrow 0 = \emptyset && \complement \boldsymbol{\alpha} = \emptyset &&
\end{aligned}
$$

Note that this representation of ideals in the case $\alpha = \omega$ is not the same as the representation for $(\mathbb{N}, \leq)$ given at the beginning of this subsection: in one case we use $\downarrow_< n$ while in the other we use $\downarrow n$.

Remark 3.2.1. *The computability conditions that the ordinal $\boldsymbol{\alpha}$ must satisfy is voluntarily vague. Notation systems for high ordinals can be complicated, and it does not fit our purpose to give a technical and lengthy analysis of this case. We will mostly use the fact that ω^2 is ideally effective in Chapter 8, for which a canonical notation system is well-known and understood: the* Cantor Normal Form *with base ω.*

3.3 Ideally Effective Constructions

One advantage of the notion of WQO is that it is preserved under many operations: Cartesian product (Dickson's Lemma, *cf.* Section 5.3), finite sequences (Higman's Lemma, *cf.* Section 6.1), finite trees (Kruskal's Theorem), finite sets (*cf.* Section 7.3), etc.

Many of the WQOs encountered in practice (in verification, graph theory, semantics, logic, ...) are actually built by incremental application of such constructions, starting from simple WQOs (essentially the ones seen in the previous section). Therefore, proving that these constructions not only preserve the property of being WQO, but also ideal effectiveness is a powerful way to prove that most of the WQOs used in practice are ideally effective. To this end, we introduce the following notion:

Definition 3.3.1. *Let C be an order-theoretic construction that preserve WQO, that is, for all WQOs $(X_1, \leq_1), \ldots, (X_n, \leq_n)$, $C[(X_1, \leq_1), \ldots, (X_n, \leq_n)]$ is a WQO (in the subsequent chapters, $n = 1$ or $n = 2$).*

Construction C is said to be ideally effective *if:*

- *It preserves ideal effectiveness, that is for every ideally effective WQOs $(X_1, \leq_1), \ldots, (X_n, \leq_n)$, $C[(X_1, \leq_1), \ldots, (X_n, \leq_n)]$ is ideally effective.*
- *A full presentation of $C[(X_1, \leq_1), \ldots, (X_n, \leq_n)]$ is computable from full presentations of the ideally effective WQOs $(X_i, \leq_i)$.*

Construction C is moreover said to be polynomial-time *if it also preserves the property of being a polynomial-time ideally effective WQO.*

Note that a construction C is said to be polynomial-time if the computed full presentation is polynomial-time. It does not indicate anything on the complexity of the function that computes this full presentation from the full presentations of the $(X_i, \leq_i)$.

In the following chapters, we prove many of the common WQO-preserving constructions to be ideally effective. This proves that most of the WQOs we use in practice are ideally effective. But it also proves that procedures to compute set-theoretic operations in these WQOs can themselves be automatically computed. Furthermore, for constructions that are not polynomial, we provide exponential lower bounds.

In the remainder of Part I, $(X, \leq)$, $(X_1, \leq_1)$, $(X_2, \leq_2)$, $(A, \leq_A)$ and $(B, \leq_B)$ designate ideally effective WQOs given by polynomial-time presentations.

Chapter 4

Generic Constructions on WQOs

Before we proceed to show that natural order-constructions are ideally effective (in the sense of Definition 3.3.1), we study three more abstract transformations that will be useful in several subsequent chapters: in the first section we ientify conditions for the extension of an ideally effective WQO to be ideally effective as well, we then study the particular case of a quotient under an equivalence relation. Finally, we identify conditions for the subset of an ideally effective WQO to be ideally effective as well.

4.1 Extension of a WQO

Let $(X, \leq)$ be a WQO and let $\leq'$ be an extension of $\leq$, as defined in Section 2.2. Then $(X, \leq')$ is also a WQO: an increasing pair for $\leq$ is in particular an increasing pair for $\leq'$. In this section, we investigate the ideals of $(X, \leq')$ and present sufficient conditions for $(X, \leq')$ to be ideally effective, assuming $(X, \leq)$ is.

4.1.1 The Ideals of $(X, \leq')$

Proposition 4.1.1. *The ideals of $(X, \leq')$ are exactly the downward closures under $\leq'$ of the ideals of $(X, \leq)$. That is,*

$$Idl(X, \leq') = \{\downarrow_{\leq'} I \mid I \in Idl(X, \leq)\}$$

Proof. ($\supseteq$): Let I be an ideal under $\leq$. Note that even though I may not be downward-closed in $(X, \leq')$, it is still directed. From there, it is easy to establish that $\downarrow_{\leq'} I$ is directed as well, non empty, and obviously downward-closed for $\leq'$. Thus it is an ideal of $(X, \leq')$.

($\subseteq$): Let J be an ideal of $(X, \leq')$. Although J may not be directed in $(X, \leq)$, it is still downward-closed under $\leq$, hence it can be decomposed as a finite union of ideals of $(X, \leq)$: $J = I_1 \cup \cdots \cup I_n$. Then $J = \downarrow_{\leq'} J = \downarrow_{\leq'} I_1 \cup \cdots \cup \downarrow_{\leq'} I_n$. Now by irreducibility of ideals, we have $J = \downarrow_{\leq'} I_i$ for some $i \in [n]$. □

Ideals of $(X, \leq')$ will thus be represented by ideals of $(X, \leq)$, and we will have to remember that I stands for $\downarrow_{\leq'} I$. Note that there might be several ideals I in $(X, \leq)$ representing the same ideal $\downarrow_{\leq'} I$ in $(X, \leq')$: the representation of an ideal may not be unique.

4.1.2 Ideal Effectiveness of an Extension

To obtain effectiveness results on $(X, \leq')$ we need to make some computability assumptions on $\leq'$. Indeed, in general, the ideal effectiveness of $(X, \leq')$ does not follow from the ideal effectiveness of $(X, \leq)$. This is witnessed by many of the examples used in Section 8.2. Some of these examples are for instance obtained by extending the natural ordering on $X = \omega^2$, which is ideally effective. In these examples, the extension is even obtained as a quotient by an equivalence relation. Hence, this legitimates the introduction of extra assumptions to prove ideal effectiveness. Chapters 7.2 and 5.4 will provide "natural" examples where these extra assumptions are not fulfilled, as shown in Sections 8.4.1 and 8.4.2.

Subsequently, we show that $(X, \leq')$ is ideally effective if we can compute closures under $\leq'$, that is we assume that the following functions are computable:

$$\begin{array}{lrl} \mathcal{C}l_{\mathrm{I}} & : \mathit{Idl}(X, \leq) & \to \mathit{Down}(X, \leq) \\ & I & \mapsto \downarrow_{\leq'} I \\ \mathcal{C}l_{\mathrm{F}} & : \mathit{Filters}(X, \leq) & \to \mathit{Up}(X, \leq) \\ & \uparrow x & \mapsto \uparrow_{\leq'}(\uparrow x) = \uparrow_{\leq'} x \end{array}$$

Note that if $I \in \mathit{Idl}(X, \leq)$, then $\downarrow_{\leq'} I$ is also downward-closed for $\leq$ and thus can be represented as a downward-closed set of $(X, \leq)$. It is precisely this representation that the function $\mathcal{C}l_{\mathrm{I}}$ outputs. Similarly for $\mathcal{C}l_{\mathrm{F}}$: $\uparrow_{\leq'} x$ is upward-closed for $\leq$. Note that using functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$, it is possible to compute the closure under $\leq'$ of arbitrary closed sets for $\leq$ using the canonical decompositions: $\downarrow_{\leq'}(I_1 \cup \cdots \cup I_n) = (\downarrow_{\leq'} I_1) \cup \cdots \cup (\downarrow_{\leq'} I_n)$ and $\uparrow_{\leq'}(\uparrow x_1 \cup \cdots \cup \uparrow x_n) = \uparrow_{\leq'} x \cup \cdots \cup \uparrow_{\leq'} x_n$.

We proceed to show that $(X, \leq')$ is ideally effective.

(XR): We use the representation of X given by the effective WQO $(X, \leq)$.

(OD): One can test $x \leq' y$ using the equivalent formulation $y \in \mathcal{C}l_{\mathrm{F}}(\uparrow_{\leq} x)$.

(IR): Ideals of $(X, \leq')$ can be represented as $\downarrow_{\leq'} I$, for ideals I of $(X, \leq)$. That is, we simply represent I, using (IR) for X, and "remember" that it stands for $\downarrow_{\leq'} I$.

(ID): Inclusion can be decided using $\mathcal{C}l_{\mathrm{I}}$ and the inclusion test for downward-closed sets of $(X, \leq)$: $\downarrow_{\leq'} I_1 \subseteq \downarrow_{\leq'} I_2$. Actually, it is sufficient to only test whether $I_1 \subseteq \downarrow_{\leq'} I_2$, calling function $\mathcal{C}l_{\mathrm{I}}$ only once, instead of twice.

Regarding ideal effectiveness:

(PI): The principal ideal $\downarrow_{\leq'} x$ of $(X, \leq')$ is represented by $\downarrow_{\leq} x$.

(CF): Given $x \in X$, $X \smallsetminus \uparrow_{\leq'} x = X \smallsetminus \mathcal{C}l_{\mathrm{F}}(\uparrow_{\leq} x)$. The latter is computable as the complement of an upward-closed set of X. The output is a downward-closed set represented by a list of ideals of $(X, \leq)$. Its canonical representation as a downward-closed set of $(X, \leq')$ is obtained using (ID) for $(X, \leq')$.

(II): Intersection of ideals is computed using $\mathcal{C}l_{\mathrm{I}}$: $\downarrow_{\leq'} I_1 \cap \downarrow_{\leq'} I_2 = \mathcal{C}l_{\mathrm{I}}(I_1) \cap \mathcal{C}l_{\mathrm{I}}(I_2)$. Here again, the result is given in terms of ideals of $(X, \leq)$, and redundancies may have to be eliminated.

(XI): If $\bigcup_i I_i = X$, then $\bigcup_i \downarrow_{\leq'} I_i = X$ as well. Thus an ideal decomposition of X is given by (XI) for $(X, \leq)$. It can be made canonical using (ID) for $(X, \leq')$, in the generic way.

Procedures for the dual operations ((CI), (IF), (XF)) are analogous.

Remark 4.1.2 (On complexity)**.** *Procedures for $(X, \leq')$ rely on procedures for functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ on the one hand, and on procedures for $(X, \leq)$ on the other hand. Thus, if $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ are computable in polynomial-time, then the complexity of operations in $(X, \leq')$ will be inherited from procedures for $(X, \leq)$ (our only consideration is whether operations are polynomial-time or not). But functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ are often costly, in which case there might be better procedures for $(X, \leq')$ than the generic ones presented in this section (*e.g. (CF) *in Section 6.2). In particular, when $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ require exponential-time computations, then so do the above procedures for $(X, \leq')$. But, we would like to point out that the cost for $\mathcal{C}l_{\mathrm{I}}$ (*resp. $\mathcal{C}l_{\mathrm{F}}$*) and operations in $(X, \leq)$ do not always compose. The following proposition proves complexity upper bounds for procedures in $(X, \leq')$ in a particular situation that will occur several times in the subsequent sections.*

Proposition 4.1.3. *Assume the following conditions:*

1. *For every $x \in X$, $\mathcal{C}l_{\mathrm{F}}(x) = \uparrow_{\leq'} x$ has at most exponentially (in $|x|$) many minimal elements for $\leq$, but these minimal elements have polynomial size in $|x|$.*

2. *Intersecting filters of $(X, \leq)$ can be performed in exponential time.*

Then, the procedure to intersect filters described in this section runs in exponential time (the naive upper bound in this case being double exponential).

Of course, this proposition can also be applied for $\mathcal{C}l_{\mathrm{I}}$ and the other operations.

Proof. We have:

$$\begin{aligned}\uparrow_{\leq'} u \cap \uparrow_{\leq'} v &= \mathcal{C}l_{\mathrm{F}}(u) \cap \mathcal{C}l_{\mathrm{F}}(v)\\ &= (\uparrow u_1 \cup \cdots \cup \uparrow u_n) \cap (\uparrow v_1 \cup \cdots \cup \uparrow v_m)\\ &= \bigcup_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} \uparrow u_i \cap \uparrow v_j\end{aligned}$$

where $(\uparrow u_1 \cup \cdots \cup \uparrow u_n)$ is the filter decomposition in $(X, \leq)$ of $\mathcal{C}l_{\mathrm{F}}(u)$ (and respectively for v), and where n and m are at most exponential in $|u|$ and $|v|$, respectively. Now, since $|u_i|$ and $|v_j|$ are polynomial in $|u|$ and $|v|$, $\uparrow u_i \cap \uparrow v_j$ can be computed in time exponential in $|u|$ and $|v|$, and finally, computing $\uparrow_{\leq'} u \cap \uparrow_{\leq'} v$ reduces to a quadratic number of exponential operations. $\square$

4.2 Quotienting under a Compatible Equivalence

In this subsection, we apply the results of this section to the most commonly encountered case of extensions: quotient under an equivalence relation. Given $(X, \leq)$ a WQO and E an equivalence relation on X such that $\leq \circ E = E \circ \leq$, define $\leq_E = \leq \circ E = E \circ \leq$. It is a relation on X, which we may see as a relation on the quotient X/E if convenient (hence the name of the section). Here, $\circ$ denotes the composition of relations, defined as follows: for all $x, y \in X$, $xR \circ Sy$ if and only if there exists z such that xRz and zSy.

The relation $\leq_E$ is clearly reflexive, and is transitive since

$$\leq_E \circ \leq_E = (\leq \circ E) \circ (\leq \circ E) = \leq \circ\ (E \circ \leq) \circ\ E = \leq \circ\ (\leq \circ\ E) \circ E = \leq \circ\ E = \leq_E$$

Moreover, $\leq_E$ is a WQO since it is an extension of $\leq$, and therefore, the results of the previous section apply. In particular, the ideals of $(X, \leq_E)$ are the subsets $\downarrow_{\leq_E} I$ for $I \in \mathit{Idl}(X, \leq)$. However, in this context, $\downarrow_{\leq_E} I = \overline{I}$, where $\overline{S}$ denotes the closure of a subset $S \subseteq X$ under E: $\overline{S} = \{y : \exists x \in S\ x\ E\ y\}$.

Proposition 4.2.1. *The ideals of $(X, \leq_E)$ are exactly the closures of the ideals of $(X, \leq)$ under E. That is,*

$$\mathit{Idl}(X, \leq_E) = \left\{\overline{I} : I \in \mathit{Idl}(X, \leq)\right\}$$

Proof. As sketched above, it suffices to show that $\downarrow_{\leq_E} I = \overline{I}$. The right-to-left inclusion is trivial. For the other direction, if $x \leq_E y$ for some $y \in I$, then there exists z such that $x\ E\ z \leq y$, and since I is downward-closed, $z \in I$ and $x \in \overline{I}$. □

Because $\leq_E$ is an extension of $\leq$, $(X, \leq_E)$ is ideally effective as soon as the functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ (introduced in the previous section) are computable. The above proof shows that in this particular setting, $\mathcal{C}l_{\mathrm{I}}(I) = \overline{I}$. We can similarly show that $\mathcal{C}l_{\mathrm{F}}(F) = \overline{F}$, but using the formulation $\leq_E = \leq \circ\ E$ instead of $\leq_E =\ E \circ \leq$. Note that because filters are always principal, $\mathcal{C}l_{\mathrm{F}}(\uparrow_{\leq_E} x) = \uparrow_{\leq} [x]_E$, where $[x]_E$ denotes the equivalence class of x.

To sum up, $(X, \leq_E)$ is ideally effective provided we can compute closures under E.

4.3 Induced WQOs

Let $(X, \leq)$ be a WQO. A subset Y of X (not necessarily finite) induces a quasi-ordering $(Y, \leq \cap\, Y \times Y)$ which is also WQO. Any subset $S \subseteq X$ induces a subset $Y \cap S$ in Y. Obviously, if S is upward-closed (or downward-closed) in X, then it induces an upward-closed (resp. downward-closed) subset in Y. However an ideal I (or a filter F) in X does not always induce an ideal (resp. a filter) in Y.

In the other direction, closed subsets of Y may of course not be closed in X. But directedness is preserved, and for any $J \in \mathit{Idl}(Y)$, the downward closure $\downarrow_X J$ is an ideal of $(X, \leq)$.

4.3.1 The ideals of $(Y, \leq)$

We say that an ideal $I \in \mathit{Idl}(X)$ is *in the adherence* of Y if $I = {\downarrow_X}(I \cap Y)$.

The next lemma proves that the ideals of Y are exactly the subsets induced by ideals of X that are in the adherence of Y.

Lemma 4.3.1. *$J \in \mathit{Idl}(Y)$ iff $J = I \cap Y$ for some $I \in \mathit{Idl}(X)$ in the adherence of Y. Furthermore, this I is unique.*

Proof. If $J \in \mathit{Idl}(Y)$ then $I \stackrel{\text{def}}{=} {\downarrow_X} J$ is an ideal of X which is in the adherence of Y, and $J = I \cap Y$.

In the other direction, if $I \in \mathit{Idl}(X)$ is in the adherence of Y then $J \stackrel{\text{def}}{=} I \cap Y$ is nonempty (otherwise ${\downarrow_X}(I \cap Y) = I$ would be) and it is directed since for any $x, y \in J$ there is $z \in I$ above x and y, and $z \leq z'$ for some $z' \in J$ since I is below Y.

Uniqueness is clear since the compatibility assumption "$I = {\downarrow_X}(I \cap Y)$" completely determines I from the ideal $J = I \cap Y$ it induces. □

An alternative definition of *adherence* often found in the literature (*e.g.* in [28, 30]) is the following one: an ideal $I \in \mathit{Idl}(X)$ is in the adherence of Y if and only if there exists a directed subset $\Delta \subseteq Y$ such that $I = {\downarrow_X} \Delta$. With this equivalence in mind, Lemma 4.3.1 extends Lemma 4.6 from [28].

Proof. of equivalence of the two notions of adherence ($\Rightarrow$) Assume $I = {\downarrow_X}(I \cap Y)$. We show that $\Delta = I \cap Y$ is directed: let $x, y \in \Delta \subseteq I$, since I is directed, there exists $z \in I$ such that $z \geq x, y$. But since $I = {\downarrow_X} \Delta$, there exists $z' \in \Delta$ such that $z' \geq z \geq x, y$, which proves that Δ is directed.

($\Leftarrow$) Assume there exists a directed subset $\Delta \subseteq Y$ such that $I = {\downarrow_X} \Delta$. Then ${\downarrow_X}(I \cap Y) = {\downarrow_X}({\downarrow_X} \Delta \cap Y) = {\downarrow_X}(\Delta \cap Y) = {\downarrow_X} \Delta = I$. □

Similarly, we can define a notion of adherence for filters. However, in this case, the condition $F = {\uparrow_X}(F \cap Y)$ for some filter $F = {\uparrow} x$ is actually equivalent to $x \in Y$ (actually to $x' \in Y$ for some $x' \equiv_X x$ when $\leq$ is not antisymmetric). Therefore, we can represent filters of Y by elements of Y, as usual.

4.3.2 Ideal Effectiveness of an Induced Quasi-Ordering

Assume that $(X, \leq)$ is an ideally effective WQO. Then $(Y, \leq)$ is not always ideally effective (even if Y is recursive). An example is given in Section 8.5. To prove that $(Y, \leq)$ is ideally effective, we assume of course that Y is recursive, but also that the two following functions are computable:

$$\begin{array}{lrl} \mathcal{S}_{\mathrm{I}} & : \mathit{Idl}(X, \leq) & \to \mathit{Down}(X, \leq) \\ & I & \mapsto {\downarrow_X}(I \cap Y) \\ \mathcal{S}_{\mathrm{F}} & : \mathit{Filters}(X, \leq) & \to \mathit{Up}(X, \leq) \\ & F & \mapsto {\uparrow_X}(F \cap Y) \end{array}$$

Under these assumptions, $(Y, \leq)$ is effective.

(XR): The subset Y is recursive by assumption, thus we can represent elements of Y as elements of X, and test whether $x \in Y$ when needed.

(OD): The quasi-ordering $\leq$ is decidable on X, thus on Y.

(IR): Ideals of Y are represented by ideals of X that are in the adherence of Y. Thus, $Idl(Y)$ is recursive since an ideal I of X is in the adherence of Y if and only if it is a fixpoint of the function $\mathcal{S}_{\mathrm{I}}$.

(ID): Finally, ideal inclusion is decidable, as shown in the next lemma.

Lemma 4.3.2. *Given I, I' two ideals of X adherent to Y, $(I \cap Y) \subseteq (I' \cap Y)$ iff $I \subseteq I'$, thus, inclusion can be directly tested within X (using* (ID)*).*

Proof. $(I \cap Y) \subseteq (I' \cap Y)$ entails $\downarrow_X(I \cap Y) \subseteq \downarrow_X(I' \cap Y)$ which, by the definition of adherence, is exactly $I \subseteq I'$. The other direction is clear. □

Before presenting procedures for all the operations on downward-closed sets, we need to be able to compute the representation of $D \cap Y$ (as a downward-closed set of Y) for a given $D \in Down(X)$.

Lemma 4.3.3. *Let $D \in Down(X)$. The canonical representation of $D \cap Y$ (as a downward-closed set of Y) is exactly the canonical representation of $\downarrow_X(D \cap Y)$ (as a downward-closed set of X).*

Observe that if $D = \bigcup_i I_i$ then $\downarrow_X(D \cap Y) = \bigcup_i \downarrow_X(I_i \cap Y) = \bigcup_i \mathcal{S}_{\mathrm{I}}(I_i)$. Thus the canonical representation of $D \cap Y$ is indeed computable.

Proof. Let $\bigcup_i I_i$ be the canonical decomposition of $\downarrow_X(D \cap Y)$. Remember that an ideal J of Y is represented by the unique ideal I of X which is in the adherence of Y such that $J = I \cap Y$. Thus, stating that $\bigcup_i I_i$ is the canonical representation of $D \cap Y$ means that:

1. $D \cap Y = \bigcup_i (I_i \cap Y)$
2. for every i, $I_i \cap Y$ is an ideal of Y
3. $I_i \cap Y$ and $I_j \cap Y$ are incomparable for inclusion, for $i \neq j$.

For the first point, $\bigcup_i (I_i \cap Y) = (\bigcup_i I_i) \cap Y = (\downarrow_X(D \cap Y)) \cap Y = D \cap Y$.

We now argue that each $I_i \cap Y$ is indeed an ideal of Y, *i.e.* all I_i's are in the adherence of Y. One inclusion being trivial, we need to show that $I_i \subseteq \downarrow_X(I_i \cap Y)$, for any i. Let $x_i \in I_i$. Since ideals I_j's are incomparable for inclusion, there exists $x'_i \in I_i$ such that $x_i \leq x'_i$ and for any $j \neq i$, $x'_i \notin I_j$ (I_i is directed). Besides, $x'_i \in I_i \subseteq \downarrow_X(D \cap Y)$ and thus there is x''_i such that $x'_i \leq x''_i \in D \cap Y$. The I_j's being downward-closed, x''_i must belong to $I_i \cap Y$, which concludes that $x_i \in \downarrow_X(I_i \cap Y)$.

Finally, the ideal decomposition $D \cap Y = \bigcup_j (I_j \cap Y)$ is canonical according to Lemma 4.3.2 and the fact that the I_j's are incomparable in X. □

The dual of Lemma 4.3.3 for upward-closed sets also holds:

Lemma 4.3.4. *Given $U \in Up(X)$, a canonical representation of $U \cap Y$ (as an upward-closed set of Y) can be computed from a canonical representation of $\uparrow_X(U \cap Y)$ (as an upward-closed set of X).*

Here also, a canonical representation of $\uparrow_X(U \cap Y)$ is computable from U using function $\mathcal{S}_{\mathrm{F}}$.

Proof. Let $\bigcup_i \uparrow x_i$ be a canonical filter decomposition (in X) of the upward-closed set $\uparrow_X(U \cap Y)$. We first prove that for every i, x_i is equivalent to some element of Y. Indeed, since $\uparrow_X x_i \subseteq \uparrow_X(U \cap Y)$, there exists $y \in U \cap Y$ with $y \leq x_i$. But then, y must be in some $\uparrow_X x_j$. Since the decomposition is canonical, the x_j's are incomparable, hence we cannot have $x_j \leq y \leq x_i$ for $j \neq i$. Thus, $x_i \equiv y \in Y$.

Moreover, we can compute a canonical filter decomposition of $\uparrow_X(U \cap Y)$ using only elements in Y: for each x_i, it is decidable whether $x_i \in Y$ (our first assumption on Y). If not, we can enumerate elements of Y until we find some $y_i \equiv x_i$. Such an element exists, and thus the enumeration terminates.

We thus obtain a canonical filter decomposition $\bigcup_i \uparrow y_i$ of $\downarrow_X(U \cap Y)$ with $y_i \in Y$. The rest of the proof is similar to the proof of Lemma 4.3.3.

□

We can now give procedures for all operations in $(Y, \leq)$:

(PI): Given $y \in Y$, the principal ideal $\downarrow_Y y$ is represented by the ideal $\downarrow_X y$, which is in the adherence of Y and computable using (PI) for X.

(XI): The set Y seen as a downward-closed set of Y is computed as the subset induced by the downward-closed set X of X, using Lemma 4.3.3 and (XI) for X (this requires $\mathcal{S}_{\mathrm{I}}$).

(XF): The set Y seen as an upward-closed set of Y is computed as the subset induced by the upward-closed set X of X, using Lemma 4.3.4 and (XF) for X.

(CF): Given $y \in Y$, the complement of $\uparrow_Y y$ is computed with $Y \smallsetminus \uparrow_Y y = (X \smallsetminus \uparrow_X y) \cap Y$. The downward-closed set $(X \smallsetminus \uparrow_X y)$ is computable using (CF) for X, and its intersection with Y is computable using Lemma 4.3.3.

(II): Given two ideals I and I' in the adherence of Y, the intersection of the ideals they induce is $(I \cap Y) \cap (I' \cap Y) = (I \cap I') \cap Y$, which is computable using (II) for X and Lemma 4.3.3.

(IF): The intersection of filters is computed as the intersection of ideals: given $y_1, y_2 \in Y$, $(\uparrow_Y y_1) \cap (\uparrow_Y y_2) = (\uparrow_X y_1 \cap \uparrow_X y_2) \cap Y$, which is computable using (IF) for X and Lemma 4.3.4.

(CI): Given an ideal I in the adherence of Y, $Y \smallsetminus (I \cap Y) = (X \smallsetminus I) \cap Y$ which is computable using (CI) for X and Lemma 4.3.4.

Remark 4.3.5. *If Y is a downward-closed subset of X, then I is adherent to Y if and only if $I \subseteq Y$, and therefore $Idl(Y) = Idl(X) \cap \mathcal{P}(Y)$. Moreover, $\mathcal{S}_{\mathrm{I}}$ is computable* *thanks to* (II)*, and* $\mathcal{S}_{\mathrm{F}}(\uparrow x) = \uparrow x$ *if* $x \in Y$*,* $\mathcal{S}_{\mathrm{F}}(\uparrow x) = \emptyset$ *otherwise. Indeed, if* $x \notin Y$*, then* $\uparrow x \cap Y = \emptyset$*.*

Similarly, if Y *is upward-closed,* $\mathcal{S}_{\mathrm{F}}$ *can be computed with* (II)*, and* $\mathcal{S}_{\mathrm{I}}(I) = I$ *if* $Y \cap I \neq \emptyset$*,* $\mathcal{S}_{\mathrm{I}}(I) = \emptyset$ *otherwise. Again,* $Y \cap I \neq \emptyset$ *if and only if* $\exists x \in \min(Y).\, x \in I$*. Given such an* x*, then* $\forall y \in I. \exists z \in I. z \geq x, y$ *by directedness. Therefore,* $I \subseteq \downarrow(I \cap \uparrow x) \subseteq \downarrow(I \cap Y)$*.*

4.4 References and Related Work

The notion of adherence has first been introduced in [28]. In this paper, the authors use the notion of ideals to "demystify" the data-structures used in the well-known but obscure proof of decidability of the reachability problem for Petri Nets first shown by Kosaraju, Lambert and Mayr. Intuitively, they define, using natural constructions on WQOs, an over-approximation of the set of all runs of a given Petri Net. In the light of this section, the ideals of the actual set of runs are the ideals of the over-approximation that are adherent to the actual set of runs, which motivated the introduction of adherence in the first place. Note that in this setting, $\mathcal{S}_{\mathrm{I}}$ is not computable.

The notion of adherence has also been successfully applied to *separability by piecewise testable languages* in language theory [30] and [31].

Chapter 5

Sums and Products of WQOs

The results of this chapter are easily obtained and widely known. They are nonetheless included here for completeness, since sums and products constitute basic constructions that naturally appear when working with WQOs (see Section 9.2 for instance). They also provide warm-up examples to train our definition.

5.1 Disjoint Sums

The *disjoint sum* $X_\sqcup = X_1 \sqcup X_2$ of two WQOs $(X_1, \leq_1)$ and $(X_2, \leq_2)$ is the set $\{1\} \times X_1 \cup \{2\} \times X_2$, well quasi-ordered by:

$$\langle i, x\rangle \leq_\sqcup \langle j, y\rangle \text{ iff } i = j \text{ and } x \leq_i y\ .$$

This construction is easily seen effective:

(XR): Elements of $X_\sqcup$ are represented as elements of $\{1\} \times X_1 \cup \{2\} \times X_2$.

(OD): The quasi-ordering is obviously decidable, when $\leq_1$ and $\leq_2$ are.

(IR): The reader may check that $I \subseteq X_\sqcup$ is an ideal if, and only if, I is some $\{i\} \times J$ for J an ideal of X_i. Thus $(Idl(X_1 \sqcup X_2), \subseteq)$ is isomorphic to $(Idl(X_1), \subseteq) \sqcup (Idl(X_2), \subseteq)$, which provides a simple data structure for $Idl(X_\sqcup)$.

(ID): Therefore, inclusion between ideals of $X_\sqcup$ reduces to inclusion between ideals of X_1 and, respectively, of X_2.

In the following, we abuse notation and, for a downward-closed subset $D = \bigcup_a I_a$ of X_i, we write $\langle i, D\rangle$ to denote $\bigcup_a \langle i, I_a\rangle$, a downward-closed subset of $X_\sqcup$ represented via ideals. Similarly, for an upward-closed subset $U = \bigcup_a \uparrow_{X_i} x_a$ of X_i, we let $\langle i, U\rangle$ denote $\bigcup_a \uparrow_\sqcup \langle i, x_a\rangle$.

We now show that $X_1 \sqcup X_2$ is ideally effective when X_1 and X_2 are. For this, we write $\bar{\imath}$ for $3 - i$ when $i \in \{1, 2\}$, so that $\{i, \bar{\imath}\} = \{1, 2\}$.

(PI): we use $\downarrow_\sqcup \langle i, x\rangle = \langle i, \downarrow_i x\rangle$ for $i \in \{1, 2\}$.

(CF): we use $X_\sqcup \smallsetminus \uparrow_\sqcup \langle i, x\rangle = \langle i, X_i \smallsetminus \uparrow_i x\rangle \cup \langle \bar{\imath}, X_{\bar{\imath}}\rangle$. Note that this relies on (CF) for X_i (to express $X_i \smallsetminus \uparrow_i x$ as a union of ideals) and on (XI) for $X_{\bar{\imath}}$.

(II): we rely on (II) for X_1 and X_2, using

$$\langle i, I\rangle \cap \langle j, J\rangle = \begin{cases} \langle i, I \cap J\rangle & \text{if } i = j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Operations (XF), (IF) and (CI) are analogous.

5.2 Lexicographic Sums

The *lexicographic sum* $X_1 + X_2$ of two WQOs $(X_1, \leq_1)$, $(X_2, \leq_2)$ has the same support set as for their disjoint sum (i.e., $X_+ \stackrel{\text{def}}{=} X_\sqcup$); only the quasi-ordering is different:

$$\langle i, x\rangle \leq_+ \langle j, y\rangle \text{ iff } i < j \text{ or } (i = j \text{ and } x \leq_i y)\ .$$

Note that $\leq_+$ extends $\leq_\sqcup$ (*i.e.* $\leq_\sqcup \subseteq \leq_+$), thus it is a WQO.

Once again, the effectiveness of this construction is trivial:

(XR): We use the same representation as before for the set $X_+ = X_\sqcup$.

(OD): The quasi-ordering is trivially decidable.

(IR): We let the reader check that $I \subseteq X_1 + X_2$ is an ideal if, and only if, I is $\{1\} \times J_1$ for some $J_1 \in \mathit{Idl}(X_1)$, or is $\{1\} \times X_1 \cup \{2\} \times J_2$ for some $J_2 \in \mathit{Idl}(X_2)$. Thus $(\mathit{Idl}(X_1{+}X_2), \subseteq)$ is isomorphic to $(\mathit{Idl}(X_1), \subseteq){+}(\mathit{Idl}(X_2), \subseteq)$, which provides a simple data structure for the set of ideals. Note that with this representation, a pair (i, J) where $J \in \mathit{Idl}(X_i)$ denotes the subset $\{1\} \times J$ when $i = 1$, and $\{1\} \times X_1 \cup \{2\} \times J$ when $i = 2$.

(ID): Ideal inclusion thus simply coincides with the quasi-ordering $\leq_+$ over $(\mathit{Idl}(X_1), \subseteq)+(\mathit{Idl}(X_2), \subseteq)$.

Subsequently, we provide a full presentation for $X_1 + X_2$, using the same abbreviations $\langle i, U\rangle$ and $\langle i, D\rangle$ we used for disjoint sums.

(PI): $\downarrow_+\langle i, x\rangle$ is (represented by) $\langle i, \downarrow_i x\rangle$.

(CF): the complement $X_+ \smallsetminus \uparrow_+\langle i, x\rangle$ is (represented by) $\langle i, X_i \smallsetminus \uparrow_i x\rangle$ except when $i = 2$ and $\uparrow_i x = X_2$, in which case $X_+ \smallsetminus \uparrow_+\langle 2, x\rangle$ is $\langle 1, X_1\rangle$.

(II): To intersect ideals, considers two cases. First $\langle 1, I\rangle \cap \langle 2, J\rangle$ is (represented by) $\langle 1, I\rangle$ for ideals issued from different components in X_+. For $\langle i, I\rangle \cap \langle i, J\rangle$, i.e., ideals issued from the same component, we use $\langle i, I \cap J\rangle$ except when $i = 2$ and $I \cap J = \emptyset$, in which case $\langle 2, I\rangle \cap \langle 2, J\rangle$ is $\langle 1, X_1\rangle$.

It is easy to implement (XI), and simple procedures for the other axioms also exist.

5.3 Cartesian Products and Dickson's Lemma

If $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are two QOs, $(X_1 \times X_2, \leq_\times)$, often just denoted $X_1 \times X_2$ or $X_\times$, is the QO defined with $\langle x_1, x_2\rangle \leq_\times \langle y_1, y_2\rangle \overset{\text{def}}{\Leftrightarrow} x_1 \leq_1 y_1 \wedge x_2 \leq_2 y_2$. Dickson's Lemma states that $(X_1 \times X_2, \leq_\times)$ is a WQO when $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are.

5.3.1 The Ideals of $(X_1 \times X_2, \leq_\times)$

Elements of $X_1 \times X_2$ are represented as pairs of elements of X_1 and X_2, denoted as $\langle x, y\rangle$ for $x \in X_1$ and $y \in X_2$ (XR). The quasi-ordering $\leq_\times$ over $X_1 \times X_2$ is trivially decidable (OD). As it is well known, the ideals of $X_1 \times X_2$ are exactly the products of ideals of X_1 and X_2.

Lemma 5.3.1. (IR)*:* $\mathit{Idl}(X_1 \times X_2) = \{I_1 \times I_2 \mid I_1 \in \mathit{Idl}(X_1), I_2 \in \mathit{Idl}(X_2)\}$.

Proof. ($\supseteq$): One checks that $I = I_1 \times I_2$ is nonempty, downward-closed, and directed, when I_1 and I_2 are. For directedness, we consider two elements $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in I$. Since I_1 is directed and contains x_1, y_1, it contains some z_1 with $x_1 \leq_1 z_1$ and $y_1 \leq_1 z_1$. Similarly I_2 contains some z_2 above x_2 and y_2 (wrt. $\leq_2$). Finally, $\langle z_1, z_2\rangle$ is in I, and above both $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$.
($\subseteq$): Consider $I \in \mathit{Idl}(X_1 \times X_2)$ and write I_1 and I_2 for its projections on X_1 and X_2. These projections are downward-closed (since I is), nonempty (since I is) and directed (since I is), hence they are ideals (in X_1 and X_2). We now show that $I_1 \times I_2 \subseteq I$. Consider an arbitrary $x_1 \in I_1$: since I_1 is the projection of I, there is some $y_2 \in X_2$ such that $\langle x_1, y_2\rangle \in I$. Similarly, for any $x_2 \in I_2$, there is some $y_1 \in X_1$ such that $\langle y_1, x_2\rangle \in I$. Since I is directed, there is some $\langle z_1, z_2\rangle \in I$ with $\langle x_1, y_2\rangle \leq_\times \langle z_1, z_2\rangle$ and $\langle y_1, x_2\rangle \leq_\times \langle z_1, z_2\rangle$. But then $x_1 \leq_1 z_1$ and $x_2 \leq_2 z_2$. Thus $\langle x_1, x_2\rangle \in I$ since I contains $\langle z_1, z_2\rangle$ and is downward-closed. Hence $I = I_1 \times I_2$ and I is a product of ideals. □

Regarding inclusion, we have $I_1 \times I_2 \subseteq J_1 \times J_2$ iff $I_1 \subseteq J_1$ and $I_2 \subseteq J_2$ (ID).

5.3.2 The Cartesian Product is Ideally Effective

Note that if D_1 and D_2 are downward-closed in X_1 and X_2 respectively, then $D_1 \times D_2$ is downward-closed in $X_1 \times X_2$, and its canonical decomposition is computable in quadratic time from the canonical decompositions of D_1 and D_2, since products distribute over unions. The same holds for upward-closed sets too.

We now provide procedures for ideal effectiveness:

(PI): $\downarrow\langle x_1, x_2\rangle = \downarrow x_1 \times \downarrow x_2$.

(II): To compute the intersection, apply $(I_1 \times I_2) \cap (I_1' \times I_2') = (I_1 \cap I_1') \times (I_2 \cap I_2')$, and distribute the product over the unions.

(CF): Similarly for complements of filters:

$$(X_1 \times X_2) \smallsetminus \uparrow_\times\langle x_1, x_2\rangle = \big[(X_1 \smallsetminus \uparrow x_1) \times X_2\big] \cup \big[(X_1 \times (X_2 \smallsetminus \uparrow x_2)\big]$$

(XI): An ideal decomposition of $X_1 \times X_2$ is easily obtained from an ideal decomposition of X_1 and X_2, as a product of downward-closed sets.

The remaining operations are obtained symmetrically. Note also that all procedures presented in this section can be performed in polynomial-time, in the sense given at the end of Chapter 3.

5.3.3 References and Related Work

The most prominent use of the product quasi-ordering in verification is with the WQO $(\mathbb{N}^k, \leq_\times)$ of vectors of natural numbers. Without explicitly mentioning the notion of ideals, they have long been used under the form of *omega-vectors* (*e.g.* [16] and successor work), that is vectors of elements of $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega\}$. In terms of ideals, we now know that $\mathbb{N}_\omega$ simply is a convenient representation for $\mathit{Idl}(\mathbb{N})$, and thus $\mathbb{N}_\omega^k$ is a convenient representation for $\mathit{Idl}(\mathbb{N}^k)$. Set-theoretic operations are not difficult to perform in $(\mathbb{N}^k, \leq_\times)$, computational description of these operations are given in [21] for instance.

It is not difficult to generalize these results to an arbitrary product of WQOs $X_1 \times X_2$. It is therefore difficult to find any reference on that matter. We can simply mention that the structure of ideals of $(X^k, \leq_\times)$ is given as a lemma to characterize the ideals of $(X^*, \leq_*)$ (see next chapter) in [12].

The results of these sections can be generalized to Noetherian spaces [13]. There, the fact that ideals of the product are products of ideals is expressed as "Sobrification commutes with finite products", and includes a reference to this more general result.

5.4 Lexicographic Product

Given $(A, \leq_A)$ and $(B, \leq_B)$ QOs, the lexicographic quasi-ordering $\leq_{\text{lex}}$ is the QO defined on $A \times B$ by $\langle a_1, b_1 \rangle \leq_{\text{lex}} \langle a_2, b_2 \rangle \overset{\text{def}}{\Leftrightarrow} a_1 <_A a_2 \vee (a_1 \equiv_A a_2 \wedge b_1 \leq_B b_2)$, where $\equiv_A \overset{\text{def}}{=} \leq_A \cap \geq_A$ and $<_A = \leq_A \setminus \equiv_A$. Moreover, if $(A, \leq_A)$ and $(B, \leq_B)$ are WQOs, then $(A \times B, \leq_{\text{lex}})$ is a WQO as well, since $\leq_\times \subseteq \leq_{\text{lex}}$. The other important property of the lexicographic quasi-ordering is that it is linear when A and B are. More precisely, the lexicographic product of two ordinals is by definition their product. Of course, $\leq_{\text{lex}}$ is decidable in polynomial-time, in the sense of Definition 3.3.1.

Since $\leq_\times \subseteq \leq_{\text{lex}}$, we are in the setting of Section 4.1: the ideal effectiveness of $(A \times B, \leq_{\text{lex}})$ comes for free provided the two following functions are computable:

$$\begin{array}{llll} \mathcal{C}l_{\mathrm{I}} & : \mathit{Idl}(A \times B, \leq_\times) & \to \mathit{Down}(A \times B, \leq_\times) \\ & I_A \times I_B & \mapsto {\downarrow_{\text{lex}}}(I_A \times I_B) \\ \mathcal{C}l_{\mathrm{F}} & : \mathit{Filters}(A \times B, \leq_\times) & \to \mathit{Up}(A \times B, \leq_\times) \\ & {\uparrow_\times}\langle a, b \rangle & \mapsto {\uparrow_{\text{lex}}}({\uparrow_\times}\langle a, b \rangle) = {\uparrow_{\leq_{\text{lex}}}}\langle a, b \rangle \end{array}$$

The expressions of these two functions in this particular case are given in the next proposition.

Proposition 5.4.1. *Given $I \in Idl(A)$, $J \in Idl(B)$, $a \in A$ and $b \in B$:*

$$\mathcal{C}l_{\mathrm{I}}(I \times J) = \begin{cases} (\downarrow x \times J) \cup (\downarrow_{<} x \times B) & \textit{when } I = \downarrow x \textit{ is a principal ideal, for some } x \in A \\ I \times B & \textit{otherwise} \end{cases}$$

$$\mathcal{C}l_{\mathrm{F}}(\uparrow\langle a, b\rangle) = \uparrow_{\times}\langle a, b\rangle \cup (\uparrow_{>} a \times B)$$

Proof. Both left-to-right inclusions are trivial. Assume $I = \downarrow x$ is a principal ideal of A, and let $\langle y, z\rangle \in \downarrow_{<} x \times B$. Since $y < x$, $\langle y, z\rangle \leq_{\mathrm{lex}} \langle x, t\rangle$ for any $t \in B$. Since J is non-empty, $\langle y, z\rangle \in \downarrow_{\mathrm{lex}} I \times J$.

On the other hand, if I is not principal, then I has no maximal element. That is, for any $y \in I$, there exists a strictly greater $x \in I$. Therefore, for any $z \in B$, $\langle y, z\rangle \leq_{\mathrm{lex}} \langle x, t\rangle$ for any $t \in B$, and in particular for some $t \in J$. Thus $\langle y, z\rangle \in \downarrow_{\mathrm{lex}} I \times J$.

The correctness of the expression for $\mathcal{C}l_{\mathrm{F}}$ is analoguous. □

Observe that $\downarrow_{<} a = \downarrow a \cap (A \setminus \uparrow a)$ and $\uparrow_{>} a = \uparrow a \cap (A \setminus \downarrow a)$ for any $a \in A$. Therefore, these expressions are computable when $(A, \leq_A)$ is ideally effective. Moreover, from the filter and ideal decomposition of B, it is simple to obtain the actual filter decomposition of $\uparrow_{>} a \times B$ and the actual ideal decomposition of $\downarrow_{<} x \times B$. However, a last obstacle remains to the computability of $\mathcal{C}l_{\mathrm{I}}$: we need to be able to decide whether a given ideal is principal. This is not an assumption listed in Definition 3.1.4, and this is actually undecidable in general. The reason we did not include this assumption as one of our axioms, in addition to the proof of the aformentionned undecidability, can be found in Section 8.4. As a consequence, function $\mathcal{C}l_{\mathrm{I}}$ may not be computable, and a concrete WQO for which $\mathcal{C}l_{\mathrm{I}}$ is not computable is provided in Section 8.4.

For a time, I thought that only the shortcut provided by Section 4.1 was made impracticable by this result, and that we could prove directly (axiom after axiom) that $(A \times B, \leq_{\mathrm{lex}})$ was ideally effective. Unfortunately, this is not the case: Dietrich Kuske found a mistake in one of my proofs, and I soon realised it couldn't be corrected. In the end, the concrete WQO for which function $\mathcal{C}l_{\mathrm{I}}$ is not computable also proves that in general, the ideal effectiveness of $(A \times B, \leq_{\mathrm{lex}})$ does not follow from the ideal effectiveness of $(A, \leq_A)$ and $(B, \leq_B)$. This is also formalised in Section 8.4.

Nonetheless, in commonly used WQOs, testing whether an ideal is principal is trivially decidable. We therefore obtain the following weaker result:

Theorem 5.4.2. *If $(A, \leq_A)$ and $(B, \leq_B)$ are ideally effective WQOs such that it is decidable whether a given ideal of $Idl(A)$ is principal, then $(A \times B, \leq_{\mathrm{lex}})$ is an ideally effective WQO.*

Furthermore, if principality of ideals of A can be tested in polynomial-time, and if $(A, \leq_A)$ and $(B, \leq_B)$ are polynomial-time, then so is $(A \times B, \leq_{\mathrm{lex}})$.

Chapter 6

Finite Sequences of WQOs

6.1 Higman's Quasi-Ordering

A QO $(X, \leq)$ is a WQO if and only if $(X^*, \leq_*)$ is a WQO as well (Higman's Lemma), where as defined in Chapter 2, X^* is the set of all finite sequences over X, and $\leq_*$ denotes the embedding quasi-ordering between sequences. We sometimes refer to $(X^*, \leq_*)$ as the Higman's extension of $(X, \leq)$ and $\leq_*$ is called Higman's quasi-ordering. Elements of X^* will simply be represented as lists of elements of X (XR). The quasi-ordering can be decided using a linear number of comparison in X by searching for the *left-most embedding* (OD).

6.1.1 The ideals of $(X^*, \leq_*)$

The product (for concatenation) of two sets of sequences $\boldsymbol{U}, \boldsymbol{V} \subseteq X^*$ is denoted $\boldsymbol{U} \cdot \boldsymbol{V} \stackrel{\text{def}}{=} \{\boldsymbol{u} \cdot \boldsymbol{v} \mid \boldsymbol{u} \in \boldsymbol{U}, \boldsymbol{v} \in \boldsymbol{V}\}$. The structure of ideals of $(X^*, \leq_*)$ is given in [12] where the following theorem is proved. An alternative proof is presented at the end of Section 6.1.2.

Theorem 6.1.1. (IR)*: The ideals of* $(X, \leq_*)$ *are exactly the finite products of* atoms $\boldsymbol{P} = \boldsymbol{A}_1 \cdots \boldsymbol{A}_n$*, where* atoms *are:*

- *any set of the form* $D^* \subseteq X^*$ *for* $D \in Down(X)$*,*
- *or any set of the form* $\downarrow_* I = \{\boldsymbol{x} \in X^* \mid x \in I\} \cup \{\epsilon\}$ *for* $I \in Idl(X)$*. Subsequently, given* $x \in X$*, we write* $\boldsymbol{x} \in X^*$ *to denote the sequence of length* 1 *whose only element is* x*. In the remainder of this chapter, and in other chapters,* $\downarrow_* I$ *will be denoted* $I + \epsilon$ *(this notation comes from regular expressions in language theory).*

As a consequence of this theorem, ideals of $(X^*, \leq_*)$ can be represented as finite sequences of atoms, atoms being themselves representable using the encodings of ideals and downward-closed sets of $(X, \leq)$ provided by one of its full presentations. Observe that the empty sequence of atoms, denoted $\boldsymbol{\epsilon}$ to avoid confusion with the empty sequence ϵ, denotes the singleton ideal $\{\epsilon\}$.

Proposition 6.1.2. (ID)*: Inclusion between ideals of* $(X^*, \leq_*)$ *can be tested using a linear number of inclusion tests between downward-closed sets of* X*, using a sort of* left-most embedding *search. The following equations implicitely describe an inductive algorithm deciding inclusion:*

1. *Atoms are compared as follows:*

$$\begin{aligned}(I_1 + \epsilon) \subseteq (I_2 + \epsilon) &\iff I_1 \subseteq I_2,\\ (I + \epsilon) \subseteq D^* &\iff I \subseteq D,\\ D_1^* \subseteq D_2^* &\iff D_1 \subseteq D_2,\\ D^* \subseteq (I + \epsilon) &\iff D = \emptyset.\end{aligned}$$

2. *for any ideal* $\boldsymbol{P}$*:* $\boldsymbol{\epsilon} \subseteq \boldsymbol{P}$*,*

3. *for any ideal* $\boldsymbol{P}$ *and atom* $\boldsymbol{A}$*:* $\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{\epsilon} \iff \boldsymbol{A} = \emptyset^* \wedge \boldsymbol{P} \subseteq \boldsymbol{\epsilon}$

4. *Finally, for any atoms* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*, and any ideals* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$*:*

   (a) *if* $\boldsymbol{A} \not\subseteq \boldsymbol{B}$ *then:*

$$\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q} \iff \boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{Q}$$

   (b) *if* $\boldsymbol{A} \subseteq \boldsymbol{B}$ *as in the first equivalence of case 1,* i.e. $\boldsymbol{A} = (I_1 + \epsilon)$, $\boldsymbol{B} = (I_2 + \epsilon)$ *for some* $I_1, I_2 \in Idl(X)$*, then:*

$$\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q} \iff \boldsymbol{P} \subseteq \boldsymbol{Q}$$

   (c) *if* $\boldsymbol{A} \subseteq \boldsymbol{B}$ *as in any of the three other equivalences of case 1, then:*

$$\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q} \iff \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q}$$

Proof. The three first cases being trivial, we concentrate on the forth one.

4a It is always true that $\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{Q} \to \boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B}\dot{\boldsymbol{Q}}$. Conversely, let $\boldsymbol{u} \cdot \boldsymbol{v} \in \boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q}$. Assuming $\boldsymbol{A} \not\subseteq \boldsymbol{B}$, there exists $\boldsymbol{w}' \in \boldsymbol{A} \setminus \boldsymbol{B}$ and by directedness, there exists $\boldsymbol{w} \in \boldsymbol{A} \setminus \boldsymbol{B}$ such that $\boldsymbol{w} \geq \boldsymbol{u}$. Now, if $\boldsymbol{A} = I + \epsilon$ for some $I \in Idl(X)$, then $\boldsymbol{w}$ is of length one, is not in $\boldsymbol{B}$ and thus $\boldsymbol{w} \cdot \boldsymbol{v}$, which is in $\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q}$ has to actually be in $\boldsymbol{Q}$. Since $\boldsymbol{Q}$ is downward-closed, $\boldsymbol{u} \cdot \boldsymbol{v} \in \boldsymbol{Q}$.

Otherwise, $\boldsymbol{A} = D^*$ for some $D \in Down(X)$. In this case, $\boldsymbol{w} \cdot \boldsymbol{w} \in \boldsymbol{A}$ and thus $\boldsymbol{wwv} \in \boldsymbol{B} \cdot \boldsymbol{Q}$. Because $\boldsymbol{w} \notin \boldsymbol{B}$, this implies $\boldsymbol{wv} \in \boldsymbol{Q}$, which again implies $\boldsymbol{uv} \in \boldsymbol{Q}$.

4b Here also, right-to-left implication is trivial. Conversely, assume $\boldsymbol{A} \cdot \boldsymbol{P} \subseteq \boldsymbol{B} \cdot \boldsymbol{Q}$ and $\boldsymbol{A} = (I_1 + \epsilon)$ and $\boldsymbol{B} = (I_2 + \epsilon)$ for some $I_1 \subseteq I_2 \in Idl(X)$. Let $\boldsymbol{u} \in \boldsymbol{P}$. Pick $x \in I_1$: $\boldsymbol{xu} \in \boldsymbol{A} \cdot \boldsymbol{P}$, thus $\boldsymbol{xu} \in \boldsymbol{B} \cdot \boldsymbol{Q}$. Therefore, $\boldsymbol{u} \in \boldsymbol{Q}$ since sequences of $\boldsymbol{B}$ have length at most one.

4c Left-to-right implication is trivial. For the other implication, decompose $\boldsymbol{P}$ in $\boldsymbol{P}_1 \cdot \boldsymbol{P}_2$ with $\boldsymbol{P}_1 \subseteq \boldsymbol{B}$ and $\boldsymbol{P}_2 \subseteq \boldsymbol{Q}$. Now observe that whenever $\boldsymbol{A} \subseteq \boldsymbol{B}$ but they are not both atoms of the form $I+\epsilon$ for some $I \in Idl(X)$, then $\boldsymbol{A} \cdot \boldsymbol{P}_1 \subseteq \boldsymbol{B}$. Therefore, $\boldsymbol{A} \cdot \boldsymbol{P}\boldsymbol{B} \cdot \boldsymbol{Q}$.

□

Note that when X is the finite alphabet $\{a, b\}$, then $(a+\epsilon)(a+b)^* = (a+b)^*$, thus the representations we use for ideals are not unique. Intuitively, the expression $(a+\epsilon)$ is subsumed by $(a+b)^*$. In general, if $\boldsymbol{A}$ is an atom and D is downward-closed in X such that $\boldsymbol{A} \subseteq D^*$, then $\boldsymbol{A}D^* = D^*\boldsymbol{A} = D^*$. Subsequently, we show that these are the only cause of non-uniqueness: avoiding such redundancies, every ideal has a unique representation (assuming unique representations for ideals of X).

Below, we use letters such as $\mathtt{A}, \mathtt{P}$, etc to denote sequences of atoms (syntax), and corresponding letters such as $\boldsymbol{A}$, $\boldsymbol{P}$, etc to denote the ideals obtained by taking the product (semantic). For example if $\mathtt{P} = (\mathtt{A}_1, \mathtt{A}_2, \ldots, \mathtt{A}_n)$, then $\boldsymbol{P} = \boldsymbol{A}_1\boldsymbol{A}_2 \ldots \boldsymbol{A}_n$. Thus it is possible to have $\mathtt{P}, \mathtt{Q}$ such that $\mathtt{P} \neq \mathtt{Q}$ but $\boldsymbol{P} = \boldsymbol{Q}$.

Definition 6.1.3. *A sequence of atoms $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ if said to be* reduced *if for all i, the following hold:*

- $\boldsymbol{A}_i \neq \{\epsilon\} = \emptyset^*$.
- *If $i+1 \leq n$ and $\boldsymbol{A}_{i+1}$ is some D^*, then $\boldsymbol{A}_i \not\subseteq \boldsymbol{A}_{i+1}$.*
- *If $i-1 \geq 1$ and $\boldsymbol{A}_{i-1}$ is some D^*, then $\boldsymbol{A}_i \not\subseteq \boldsymbol{A}_{i-1}$.*

Every ideal has a reduced decomposition into atoms, since any decomposition can be converted to a reduced one by dropping atoms which are redundant as per Definition 6.1.3. It remains to show uniqueness of the reduced representation:

Theorem 6.1.4. *If $\mathtt{P}$ and $\mathtt{Q}$ are reduced sequences of atoms such that $\boldsymbol{P} = \boldsymbol{Q}$, then* $\mathtt{P} = \mathtt{Q}$.

Proof. By induction on $|\mathtt{P}| + |\mathtt{Q}|$. The result is trivial if $|\mathtt{P}| \leq 1$ or $|\mathtt{Q}| \leq 1$.

Otherwise, $\mathtt{P} = \mathtt{A}_1\mathtt{A}_2\mathtt{P}'$ and $\mathtt{Q} = \mathtt{B}_1\mathtt{B}_2\mathtt{Q}'$. We first show $\boldsymbol{A}_1 = \boldsymbol{B}_1$. Seeking for a contradiction, assume $\boldsymbol{A}_1 \neq \boldsymbol{B}_1$, and without loss of generality, $\boldsymbol{A}_1 \not\subseteq \boldsymbol{B}_1$. Then since $\boldsymbol{P} \subseteq \boldsymbol{Q}$, the inclusion test described in Proposition 6.1.2 gives $\boldsymbol{P} \subseteq \boldsymbol{B}_2\boldsymbol{Q}' \subseteq \boldsymbol{B}_1\boldsymbol{B}_2\boldsymbol{Q}' \subseteq \boldsymbol{P}$, from which we deduce that $\boldsymbol{B}_2\boldsymbol{Q}' = \boldsymbol{B}_1\boldsymbol{B}_2\boldsymbol{Q}'$. The induction hypothesis then implies $\mathtt{B}_2\mathtt{Q}' = \mathtt{B}_1\mathtt{B}_2\mathtt{Q}'$, which is absurd. Thus $\boldsymbol{A}_1 = \boldsymbol{B}_1$ (and $\mathtt{A}_1 = \mathtt{B}_1$).

We now want to show that $\boldsymbol{A}_2\boldsymbol{P}' = \boldsymbol{B}_2\boldsymbol{Q}'$. Since the situation is symmetric, we only prove one inclusion. We distinguish two cases.

1. If $\boldsymbol{A}_1$ is of the form $I+\epsilon$, then so is $\boldsymbol{B}_1$ and Proposition 6.1.2 directly implies $\boldsymbol{A}_2\boldsymbol{P}' \subseteq \boldsymbol{B}_2\boldsymbol{Q}'$.
2. Otherwise, $\boldsymbol{A}_1$ is of the form D^*, in which case $\boldsymbol{A}_2 \not\subseteq \boldsymbol{A}_1 = \boldsymbol{B}_1$, because $\mathtt{P}$ is reduced. Therefore, by Proposition 6.1.2, $\boldsymbol{A}_2\boldsymbol{P}' \subseteq \boldsymbol{B}_2\boldsymbol{Q}'$.

By symmetry, we obtain $\boldsymbol{A}_2\boldsymbol{P}' = \boldsymbol{B}_2\boldsymbol{Q}'$ and conclude the proof with the induction hypothesis. □

6.1.2 The Higman Extension is Ideally Effective

In this section we proceed to show that $(X^*, \leq_*)$ is ideally effective. To this end, we will allow ourselves to write expressions $D + \epsilon$ for downward-closed sets $D \in Down(X)$ (instead of ideals). Products of such *generalized atoms* are not ideals, but they are downward-closed sets of X^*. Moreover, decomposing downward-closed sets $D \in Down(X)$ as a finite union of ideals $D = \bigcup_i I_i$ (in $(X, \leq)$), generalized atoms can be written as unions of actual atoms: $D + \epsilon = \bigcup_i (I + \epsilon)$. Therefore, distributing products over the unions, the ideal decomposition of finite products of generalized atoms $\boldsymbol{P}$ can be computed. However, the decomposition might be of size exponential in the size of $\boldsymbol{P}$.

We will also use shortcuts to denote upward-closed sets. Given two upward-closed sets of sequences $\boldsymbol{U} = \bigcup_i \uparrow \boldsymbol{u}_i$ and $\boldsymbol{V} = \bigcup_j \uparrow \boldsymbol{v}_j$, the product $\boldsymbol{U} \cdot \boldsymbol{V}$ is upward-closed and its filter decomposition can be computed by $\boldsymbol{U} \cdot \boldsymbol{V} = \bigcup_{i,j} \uparrow (\boldsymbol{u}_i \cdot \boldsymbol{v}_j)$. Moreover, if $U = \bigcup_i \uparrow_X x \in Up(X)$ is an upward-closed set of $(X, \leq)$, then we can see U as a set of sequences of length 1, and $\uparrow_* U = \bigcup_i \uparrow_* \boldsymbol{x}_i$. Therefore, an upward-closed set of X can always be understood as an upward-closed set of X^*. This will be used silently.

The axioms (XF), (XI) and (PI) are trivially satisfied: $X^* = \uparrow \epsilon$, X^* is an atom, and $\downarrow (x_1 \cdots x_n) = (\downarrow x_1 + \epsilon) \cdots (\downarrow x_n + \epsilon)$. Also, to test ideal membership $\boldsymbol{u} \in \boldsymbol{P}$, we use $\downarrow \boldsymbol{u} \subseteq \boldsymbol{P}$. In the four subsequent subsections, we give procedures to complement filters, intersect ideals, intersect filters and lastly complement ideals, thus establishing that $(X^*, \leq_*)$ is ideally effective. Note that these procedures are uniform in $(X, \leq)$, therefore the Higman extension of a WQO is an ideally effective operation. However, we will prove that it is not polynomial-time (in the sense of Definition 3.1.4).

In the last two subsections, we give a proof of Theorem 6.1.1, and provide some references and pointers to related work.

Complementing Filters (CF)

Proposition 6.1.5. *Given $\boldsymbol{w} \in X^*$, the downward-closed set $X^* \smallsetminus \uparrow \boldsymbol{w}$ can be computed inductively using the following equations:*

$$X^* \smallsetminus \uparrow \epsilon = \emptyset \text{ (empty union)} \tag{6.1}$$

$$X^* \smallsetminus \uparrow \boldsymbol{x} = (X \smallsetminus \uparrow x)^* \tag{6.2}$$

$$X^* \smallsetminus \uparrow \boldsymbol{x}\boldsymbol{w} = (X \smallsetminus \uparrow x)^* \cdot (X + \epsilon) \cdot (X^* \smallsetminus \uparrow \boldsymbol{w}) \tag{6.3}$$

Note that X might not be an ideal, and thus $X + \epsilon$ might be a generalized atom (not an actual atom). In this case, to compute the actual ideal decomposition of $X^* \smallsetminus \uparrow \boldsymbol{w}$, we need to distribute the products over the unions that comes from the ideal decomposition of X. This step might result in an exponential blow up. For instance, take X to be the finite ordering that consists of three elements $0, 1$ and $1'$ with $0 < 1$, $0 < 1'$ and $1 \perp 1'$. Then $X^* \smallsetminus \uparrow 0^{n+1}$ consists of all sequences of length at most n. Thus its canonical ideal decomposition has size 2^n: $X^* \smallsetminus \uparrow 0^{n+1} = \bigcup_{\boldsymbol{u} \in \{1,1'\}^n} \downarrow \boldsymbol{u}$.

Note however that in the commonly encountered case where X is a finite alphabet, the operation of complementing filters can be performed in polynomial-time. Indeed, if $X = \{a_1, \dots, a_n\}$ is a finite alphabet (*i.e.* ordered with equality), then $X = \bigcup_{i=1}^n \downarrow a_i$, which is not an ideal for $n > 1$. But since $X \setminus \uparrow a_i = \bigcup_{j \neq i} \downarrow a_j$, $(X \setminus \uparrow a_i)^* \cdot (X + \epsilon) = (X \setminus \uparrow a_i)^* \cdot (a_i + \epsilon)$ which is an ideal. In [12], the authors prove this finer expression to complement filters. for an arbitrary WQO $(X, \leq)$:

$$X^* \setminus \uparrow x y \boldsymbol{w} = (X \setminus \uparrow x)^* \cdot [\downarrow(\uparrow x \cap \uparrow y) + \epsilon] \cdot (X^* \setminus \uparrow y \boldsymbol{w})$$

In general, our setting does not guarantees that the expression $\downarrow U$ is computable for $U \in Up(X)$, but when X is a finite alphabet, the expression $(\uparrow x \cap \uparrow y)$ either denotes the empty set or $(x + \epsilon)$ when $x = y$. Therefore, using this expression, one directly obtains the canonical form of the complement of a filter of X^* in the case of a finite alphabet.

Proof. (of Proposition 6.1.5)
Equations 6.1 and 6.2 are obvious. For the third equation: $(\supseteq)$ If $\boldsymbol{w}' = \boldsymbol{u} y \boldsymbol{v}$ with $\boldsymbol{u} \in (X \setminus \uparrow x)^*$, $y \in X + \epsilon$ and $\boldsymbol{v} \in (X^* \setminus \uparrow \boldsymbol{w})$, then assume $x\boldsymbol{w} \leq \boldsymbol{w}'$, then either $x = y$ and $\boldsymbol{w} \leq \boldsymbol{v}$ which is a contradiction, or $x\boldsymbol{w} \leq \boldsymbol{v}$, which is also a contradiction.

$(\subseteq)$ Let $\boldsymbol{w}' \notin \uparrow x\boldsymbol{w}$. Then either $\boldsymbol{w}' \in (X \setminus \uparrow x)^*$, or we can write $\boldsymbol{w}' = \boldsymbol{u} y \boldsymbol{v}$ with $\boldsymbol{u} \in (X \setminus \uparrow x)^*$ and $y \geq x$. Moreover, $\boldsymbol{v} \notin \uparrow \boldsymbol{w}$, since otherwise $x\boldsymbol{w} \leq y\boldsymbol{v} \leq \boldsymbol{u} y \boldsymbol{v} = \boldsymbol{w}'$. Therefore, $\boldsymbol{w}' \in (X \setminus \uparrow x)^* \cdot (X + \epsilon) \cdot (X^* \setminus \uparrow \boldsymbol{w})$. □

Intersecting Ideals (II)

Proposition 6.1.6. *The intersection of two ideals of $(X^*, \leq_*)$ can be computed inductively using the following equations:*

$$\begin{aligned}
\boldsymbol{\epsilon} \cap \boldsymbol{Q} &= \boldsymbol{\epsilon} \\
\boldsymbol{P} \cap \boldsymbol{\epsilon} &= \boldsymbol{\epsilon} \\
D_1^* \cdot \boldsymbol{P} \cap D_2^* \cdot \boldsymbol{Q} &= (D_1 \cap D_2)^* \cdot \big[((D_1^* \cdot \boldsymbol{P}) \cap \boldsymbol{Q}) \cup (\boldsymbol{P} \cap (D_2^* \cdot \boldsymbol{Q}))\big] \\
(I_1 + \epsilon) \cdot \boldsymbol{P} \cap (I_2 + \epsilon) \cdot \boldsymbol{Q} &= \big[((I_1 + \epsilon) \cdot \boldsymbol{P}) \cap \boldsymbol{Q}\big] \cup \big[\boldsymbol{P} \cap ((I_2 + \epsilon) \cdot \boldsymbol{Q})\big] \cup \\
&\qquad \cup \big[\big((I_1 \cap I_2) + \epsilon\big) \cdot (\boldsymbol{P} \cap \boldsymbol{Q})\big] \\
D^* \cdot \boldsymbol{P} \cap (I + \epsilon) \cdot \boldsymbol{Q} &= \big[\boldsymbol{P} \cap ((I + \epsilon) \cdot \boldsymbol{Q})\big] \cup \big[\big((D \cap I) + \epsilon\big) \cdot (D^* \cdot \boldsymbol{P} \cap \boldsymbol{Q})\big]
\end{aligned}$$

Note that, in addition to using generalized atoms ($I_1 \cap I_2$ and $D \cap I$ may not be ideals), we use expressions that mix unions and products. The actual ideal decomposition is obtained when distributing the products over the unions. This is computable, but may again result in an exponential blow-up of the ideal decomposition. This is witnessed by the following example: take $X = \{a, b\}$ a two-symbol alphabet and $D = \downarrow(aba)^n \cap \downarrow(bab)^n$. Every word $\boldsymbol{u}$ in $\{ab, ba\}^n$ is a maximal element of D: membership is obvious, and maximality can be proved using the number of symbols a and b in $\boldsymbol{u}$, $\boldsymbol{u}$ has as many a's as $(bab)^n$ and as many b's as $(aba)^n$. Therefore, D is the union of exponentially many incomparable ideals (words of the same size are either equal or incomparable).

Proof. (of Proposition 6.1.6)
The first two equations are obviously correct. The other right-to-left inclusions are easily checked using Proposition 6.1.2. For the left-to-right directions:

- Let $\boldsymbol{u} \in (D_1^* \cdot \boldsymbol{P} \cap D_2^* \cdot \boldsymbol{Q})$. Let $\boldsymbol{v}$ be the longest prefix of $\boldsymbol{u}$ which is in D_1^*. Without loss of generality, we assume that the longest prefix of $\boldsymbol{u}$ which is in D_2^* is longer than $|\boldsymbol{v}|$, and thus can be written $\boldsymbol{vw}$ for some $\boldsymbol{w} \in X^*$. Moreover, there exists $\boldsymbol{t} \in X^*$ so that $\boldsymbol{u} = \boldsymbol{vwt}$. We have, $\boldsymbol{v} \in (D_1 \cap D_2)^*$, $\boldsymbol{wt} \in \boldsymbol{P}$ and $\boldsymbol{t} \in \boldsymbol{Q}$. Therefore, $\boldsymbol{wt} \in \boldsymbol{P} \cap D_2^* \boldsymbol{Q}$.
- Let $x \cdot \boldsymbol{u} \in (I_1 + \epsilon) \cdot \boldsymbol{P} \cap (I_2 + \epsilon) \cdot \boldsymbol{Q}$. Depending on whether $x \in I_1 \smallsetminus I_2$, $x \in I_2 \smallsetminus I_1$ or $x \in I_1 \cap I_2$, $\boldsymbol{u}$ is easily proved to be in one of the three sets it should belong to. In the case x is neither in I_1 nor I_2, then $\boldsymbol{u}$ belongs to all three sets.
- The last case is proved similarly. □

Intersecting Filters (IF)

Proposition 6.1.7. *The intersection of two filters can be computed inductively using the following equations:*

$$\uparrow \boldsymbol{v} \cap \uparrow \epsilon = \uparrow \boldsymbol{v}$$
$$\uparrow \epsilon \cap \uparrow \boldsymbol{w} = \uparrow \boldsymbol{w}$$
$$\uparrow \boldsymbol{xv} \cap \uparrow \boldsymbol{yw} = \big[(\uparrow \boldsymbol{x}) \cdot (\uparrow \boldsymbol{v} \cap \uparrow \boldsymbol{yw})\big] \cup \big[(\uparrow \boldsymbol{y}) \cdot (\uparrow \boldsymbol{xv} \cap \uparrow \boldsymbol{w})\big] \cup \big[(\uparrow_X x \cap \uparrow_X y) \cdot (\uparrow \boldsymbol{v} \cap \uparrow \boldsymbol{w})\big]$$

where $\boldsymbol{v}, \boldsymbol{w} \in X^*$ and $a, b \in X$. The actual filter decomposition of this last upward-closed set can be computed following the remark at the begining of this section.

Proof. The first two equations are trivial. For the third, right-to-left inclusion is obvious. Left-to-right inclusion: consider $\boldsymbol{u} \in \uparrow a\boldsymbol{v} \cap \uparrow b\boldsymbol{w}$. Let $\boldsymbol{u}'$ be the shortest suffix of $\boldsymbol{u}$ which belongs to $\uparrow a\boldsymbol{v} \cap \uparrow b\boldsymbol{w}$. Consider cases depending on whether the first letter of $\boldsymbol{u}'$ is above a and b in X. □

The naive implementation of this procedure runs in exponential-time, and it is asymptotically optimal: the canonical decomposition of the upward-closed set $\uparrow a^n \cap \uparrow b^n$ has at least $\binom{2n}{n}$ minimal elements (this corresponds to all the words in the shuffle product $a^n \sqcup\!\sqcup\, b^n$).

Complementing Ideals (CI)

In this subsection we present a procedure to complement ideals. We first show how to complement atoms, and then how to complement products of atoms.

- If $D \subseteq X$ is downward-closed, then $X^* \smallsetminus D^*$ consists of all words which have at least one letter not in D. That is, compute $X \smallsetminus D = \uparrow a_1 \cup \ldots \cup \uparrow a_n$, using (CI) for X. Then $X^* \smallsetminus D^* = \uparrow_{X^*} \boldsymbol{a}_1 + \ldots + \uparrow_{X^*} \boldsymbol{a}_n$.
- If $I \subseteq X$ is an ideal, then $X^* \smallsetminus (I + \epsilon)$ consists of all words which have at least one letter not in I, and all words with at least two letters. The former is obtained as in the previous case. The latter is $\uparrow_{X^*} X \cdot X$, easily computed in a similar way using (XF) for X.

We now consider products $\boldsymbol{A}_1 \cdots \boldsymbol{A}_n$ of atoms. We know how to compute $\boldsymbol{U}_i = \complement \boldsymbol{A}_i$. We thus use the relation $\complement(\boldsymbol{A}_1 \cdots \boldsymbol{A}_n) = \complement(\complement \boldsymbol{U}_1 \cdots \complement \boldsymbol{U}_n)$, which motivates the following definition:

Definition 6.1.8. *Define the operator* $\odot : Up(X^*) \times Up(X^*) \to Up(X^*)$ *as* $\boldsymbol{U} \odot \boldsymbol{V} := \complement(\complement \boldsymbol{U} \cdot \complement \boldsymbol{V})$.

Note that $\boldsymbol{U} \odot \boldsymbol{V}$ is obviously upward-closed when $\boldsymbol{U}$ and $\boldsymbol{V}$ are. The operation $\odot$ is easily shown associative using the associativity of the product, thus $\boldsymbol{U}_1 \odot \ldots \odot \boldsymbol{U}_n = \complement(\complement \boldsymbol{U}_1 \cdot \ldots \cdot \complement \boldsymbol{U}_n)$. The previous relation becomes $\complement(\boldsymbol{A}_1 \cdots \boldsymbol{A}_n) = \boldsymbol{U}_1 \odot \cdots \odot \boldsymbol{U}_n$, and it only remains to show that $\odot$ is computable on upward-closed sets. In what follows, we will often use the following obvious characterization: $\boldsymbol{w} \in \boldsymbol{S} \odot \boldsymbol{T}$ if and only if for all factorizations $\boldsymbol{w} = \boldsymbol{w}_1 \boldsymbol{w}_2$, $\boldsymbol{w}_1 \in \boldsymbol{S}$ or $\boldsymbol{w}_2 \in \boldsymbol{T}$.

We first show that $\odot$ is computable on principal filters, and later show that $\odot$ distributes over unions.

Lemma 6.1.9. *On principal filters,* $\odot$ *can be computed using the following equations:*

$$\begin{aligned} \uparrow \boldsymbol{v} \odot \uparrow \epsilon &= X^* \\ \uparrow \epsilon \odot \uparrow \boldsymbol{w} &= X^* \\ \uparrow \boldsymbol{v}a \odot \uparrow b\boldsymbol{w} &= \uparrow(\boldsymbol{v}ab\boldsymbol{w}) \cup (\uparrow \boldsymbol{v}) \cdot (\uparrow_X a \cap \uparrow_X b) \cdot (\uparrow \boldsymbol{w}) \end{aligned}$$

where $\boldsymbol{v}, \boldsymbol{w} \in X^*$ *and* $a, b \in X$.

Proof. The first two equations are obvious. For the third:

($\supseteq$) If $\boldsymbol{u} \geq_* \boldsymbol{v}ab\boldsymbol{w}$, then for every factorization of $\boldsymbol{u} = \boldsymbol{u}_1 \boldsymbol{u}_2$, the left factor $\boldsymbol{u}_1$ is above $\boldsymbol{v}a$, or the right factor $\boldsymbol{u}_2$ is above $b\boldsymbol{w}$, and thus $\boldsymbol{u} \in \uparrow \boldsymbol{v}a \odot \uparrow b\boldsymbol{w}$. If $\boldsymbol{u} \geq_* \boldsymbol{v}c\boldsymbol{w}$, where $c \in X$ is such that $c \geq a$ and $c \geq b$, then in every factorization of $\boldsymbol{u} = \boldsymbol{u}_1 \boldsymbol{u}_2$, c appears either in the left factor $\boldsymbol{u}_1$ or in the right factor $\boldsymbol{u}_2$, and this suffices to show that either $\boldsymbol{u} \geq_* \boldsymbol{v}a$ or $\boldsymbol{u} \geq_* b\boldsymbol{w}$.

($\subseteq$) Let $\boldsymbol{u} \in (\uparrow \boldsymbol{v}) \odot (\uparrow \boldsymbol{w})$. From the factorizations $\boldsymbol{u} = \boldsymbol{u} \cdot \epsilon$ and $\boldsymbol{u} = \epsilon \cdot z$ we get $\boldsymbol{v}a \leq_* \boldsymbol{u}$ and $b\boldsymbol{w} \leq_* \boldsymbol{u}$. Consider the shortest prefix of $\boldsymbol{u}$ into which $\boldsymbol{v}a$ embeds and the shortest suffix into which $b\boldsymbol{w}$ embeds. If these factors don't overlap, we get $\boldsymbol{u} \geq_* \boldsymbol{v}ab\boldsymbol{w}$. If they overlap, the overlap must have length exactly one (otherwise $\boldsymbol{u}$ can be split anywhere in the middle of the overlap to obtain a contradiction). Write $\boldsymbol{u} = \boldsymbol{u}_1 c \boldsymbol{u}_2$ where $c \in X$ is the overlap. Then $\boldsymbol{u}_1 \geq_* \boldsymbol{v}$, $c \geq a$, $c \geq b$, and $\boldsymbol{u}_2 \geq_* \boldsymbol{w}$, which proves the statement. ☐

It now remains to show that $\odot$ distributes over unions.

Lemma 6.1.10. *Given* $\boldsymbol{U}, \boldsymbol{U}_1, \boldsymbol{U}_2$ *three upward-closed sets of* X^**, we have:*

$$\begin{aligned} (\boldsymbol{U}_1 \cup \boldsymbol{U}_2) \odot \boldsymbol{U} &= (\boldsymbol{U}_1 \odot \boldsymbol{U}) \cup (\boldsymbol{U}_2 \odot \boldsymbol{U}) \\ \boldsymbol{U} \odot (\boldsymbol{U}_1 \cup \boldsymbol{U}_2) &= (\boldsymbol{U} \odot \boldsymbol{U}_1) \cup (\boldsymbol{U} \odot \boldsymbol{U}_2) \end{aligned}$$

Proof. We actually show the equivalent statement that product distributes over intersection for downward-closed sets. Let $\boldsymbol{D} = \complement \boldsymbol{U}$, $\boldsymbol{D}_1 = \complement \boldsymbol{U}_1$ and $\boldsymbol{D}_2 = \complement \boldsymbol{U}_2$, we show that $(\boldsymbol{D}_1 \cap \boldsymbol{D}_2) \cdot \boldsymbol{D} = (\boldsymbol{D}_1 \cdot \boldsymbol{D}) \cap (\boldsymbol{D}_2 \cdot \boldsymbol{D})$, and $\boldsymbol{D} \cdot (\boldsymbol{D}_1 \cap \boldsymbol{D}_2) = (\boldsymbol{D} \cdot \boldsymbol{D}_1) \cap (\boldsymbol{D} \cdot \boldsymbol{D}_2)$.

We only show the first equation, the second one being symmetrical. The left-to-right inclusion is obvious. For the right-to-left inclusion, let $\boldsymbol{w} \in \boldsymbol{D}_1 \cdot \boldsymbol{D} \cap \boldsymbol{D}_2 \cdot \boldsymbol{D}$. Then $\boldsymbol{w} = \boldsymbol{u}_1\boldsymbol{v}_1$ for some $\boldsymbol{u}_1 \in \boldsymbol{D}_1$ and $\boldsymbol{v}_1 \in \boldsymbol{D}$. Also, $\boldsymbol{w} = \boldsymbol{u}_2\boldsymbol{v}_2$ for some $\boldsymbol{u}_2 \in \boldsymbol{D}_2$ and $\boldsymbol{v}_2 \in \boldsymbol{D}$. One of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is a prefix of the other. Assume $\boldsymbol{u}_1$ is a prefix of $\boldsymbol{u}_2$ (the other case is analogous). Since $\boldsymbol{D}_2$ is downward-closed and $\boldsymbol{u}_1 \leq_* \boldsymbol{u}_2$, $\boldsymbol{u}_1 \in \boldsymbol{D}_2$. Thus, $\boldsymbol{u}_1 \in \boldsymbol{D}_1 \cap \boldsymbol{D}_2$ and $\boldsymbol{v}_1 \in \boldsymbol{D}$, $\boldsymbol{w} = \boldsymbol{u}_1\boldsymbol{v}_1 \in (\boldsymbol{D}_1 \cap \boldsymbol{D}_2) \cdot \boldsymbol{D}$. □

It follows that computing $\complement \boldsymbol{A}_1 \odot \cdots \odot \complement \boldsymbol{A}_n = (U_1 \odot \cdots \odot U_n)$ reduces to computing $\odot$ on filters, which is computable by Lemma 6.1.9. However, distributing over the unions can once again lead to an exponential blow-up. This is unavoidable, as shown below.

Proposition 6.1.11. *The upward-closed set $\complement \downarrow (ab)^n$ where $X = \{a, b\}$ is a two-symbol alphabet has an exponential number (in n) of minimal elements.*

Thus, ideals of X^* cannot be complemented in polynomial-time, even in the simple case where X is a two-symbol alphabet.

Proof. Since $\odot$ is associative, $\complement \downarrow (ab)^n = (\complement \downarrow (ab)^{n-1}) \odot (\complement \downarrow ab) = (\complement \downarrow (ab)^{n-1}) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa)$. This identity suggests to proceed by induction.

Let $\bigcup_{i=1}^{m} \uparrow \boldsymbol{u}_i$ be the canonical decomposition of $\complement \downarrow (ab)^n$. Define the subsets of indexes $S_{\boldsymbol{v}} = \{i \in [m] \mid \boldsymbol{v} \text{ is a suffix of } \boldsymbol{u}_i\}$.

$$
\begin{aligned}
\complement \downarrow (ab)^{n+1} = & (\complement \downarrow (ab)^n) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa) \\
= & (\bigcup_{i=1}^{m} \uparrow \boldsymbol{u}_i) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa) \\
= & (\bigcup_{i \in S_{aa}} \uparrow \boldsymbol{u}_i) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa) \cup \\
& \quad (\bigcup_{i \in S_{ba}} \uparrow \boldsymbol{u}_i) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa) \cup \\
& \quad\quad (\bigcup_{i \in S_{b}} \uparrow b\boldsymbol{u}_i) \odot (\uparrow ba \cup \uparrow bb \cup \uparrow aa) \\
= & (\bigcup_{i \in S_{aa}} \uparrow \boldsymbol{u}_i ba \cup \uparrow \boldsymbol{u}_i bb \cup \uparrow \boldsymbol{u}_i a) \cup \\
& \quad (\bigcup_{i \in S_{ba}} \uparrow \boldsymbol{u}_i ba \cup \uparrow \boldsymbol{u}_i bb \cup \uparrow \boldsymbol{u}_i a) \cup \\
& \quad\quad (\bigcup_{i \in S_{b}} \uparrow \boldsymbol{u}_i a \cup \uparrow \boldsymbol{u}_i b \cup \uparrow \boldsymbol{u}_i aa)
\end{aligned}
$$

This last step follow from the particular form that Lemma 6.1.9 takes when $X = A$ a finite alphabet: indeed, for $a, b \in A$, either $a \neq b$ and $\uparrow_A a \cap \uparrow_A b = \emptyset$ which means $\uparrow \boldsymbol{v}a \odot \uparrow b\boldsymbol{w} = \uparrow \boldsymbol{v}ab\boldsymbol{w}$; or $a = b$ in which case $\uparrow \boldsymbol{v}a \odot \uparrow a\boldsymbol{w} = \uparrow \boldsymbol{v}aa\boldsymbol{w} \cup \uparrow \boldsymbol{v}a\boldsymbol{w} = \uparrow \boldsymbol{v}a\boldsymbol{w}$ since $\boldsymbol{v}a\boldsymbol{w} \leq \boldsymbol{v}aa\boldsymbol{w}$.

Now, observe that for $i \in S_{aa} \cup S_{ba}$, $\boldsymbol{u}_i a \leq \boldsymbol{u}_i ba$; and for $i \in S_b$, $\boldsymbol{u}_i a \leq \boldsymbol{u}_i aa$, thus we obtain:

$$\complement{\downarrow}(ab)^{n+1} = (\bigcup_{i\in S_{aa}} \uparrow \boldsymbol{u}_i bb \cup \uparrow \boldsymbol{u}_i a) \cup (\bigcup_{i\in S_{ba}} \uparrow \boldsymbol{u}_i bb \cup \uparrow \boldsymbol{u}_i a) \cup (\bigcup_{i\in S_b} \uparrow \boldsymbol{u}_i a \cup \uparrow \boldsymbol{u}_i b) \tag{6.4}$$

There is one last less obvious redundancy: for $n \geq 2$ and $i \in S_{ba}$, there exists $\boldsymbol{u}'_i$ in the canonical decomposition of $\complement(\downarrow(ab)^{n-1})$ such that $\boldsymbol{u}_i = \boldsymbol{u}'_i a$. Indeed, we see in Equation 6.4 that a's can only be added one by one. Thus, $\boldsymbol{u}'_i$ having b as a suffix, there exists $j \in [m]$ such that $\boldsymbol{u}_j = \boldsymbol{u}'_i b$. This entails $j \in S_b$, and thus $\boldsymbol{u}_j b = \boldsymbol{u}'_i bb$ appears in the decomposition of $\complement{\downarrow}(ab)^{n+1}$. Since $\boldsymbol{u}'_i bb \leq \boldsymbol{u}'_i abb = \boldsymbol{u}_i bb$, the element $\boldsymbol{u}_i bb$ is not minimal in the decomposition and can be removed. We finally obtain:

$$\complement{\downarrow}(ab)^{n+1} = (\bigcup_{i\in S_{aa}} \uparrow \boldsymbol{u}_i bb \cup \uparrow \boldsymbol{u}_i a) \cup (\bigcup_{i\in S_{ba}} \uparrow \boldsymbol{u}_i a) \cup (\bigcup_{i\in S_b} \uparrow \boldsymbol{u}_i a \cup \uparrow \boldsymbol{u}_i b) \tag{6.5}$$

(6.6)

We now prove that the decomposition above is indeed minimal. First, $\boldsymbol{u}_i bb \leq \boldsymbol{u}_j a$ or $\boldsymbol{u}_i bb \leq \boldsymbol{u}_j b$ imply $\boldsymbol{u}_i b \leq \boldsymbol{u}_j$ which is impossible since the decomposition $\bigcup_{i=1}^m u_i$ is canonical, *i.e.* the u_i's are incomparable. Moreover, $\boldsymbol{u}_i bb \leq \boldsymbol{u}_j bb$ implies $\boldsymbol{u}_i \leq \boldsymbol{u}_j$ which implies $i = j$ since $\bigcup_{i=1}^m \uparrow \boldsymbol{u}_i$ is the canonical decomposition of $\complement{\downarrow}(ab)^n$. Similarly, $\boldsymbol{u}_i a \leq \boldsymbol{u}_j b$ and $\boldsymbol{u}_i a \leq \boldsymbol{u}_j bb$ are impossible, and $\boldsymbol{u}_i a \leq \boldsymbol{u}_j a$ implies $i = j$. The only non trivial case is when $\boldsymbol{u}_i b \leq \boldsymbol{u}_j bb$. This implies that $i \in S_b$ and $j \in S_{aa}$, and of course that $\boldsymbol{u}_i \leq \boldsymbol{u}_j b$. Again, assume $n \geq 2$, and denote by $\boldsymbol{u}'_i$ and $\boldsymbol{u}'_j$ the minimal elements of $\complement{\downarrow}(ab)^{n-1}$ from which $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ are respectively built. Then, $\boldsymbol{u}_j = \boldsymbol{u}'_j a$ and a is a suffix of $\boldsymbol{u}'_j$. As for $\boldsymbol{u}_i$, it is either equal to $\boldsymbol{u}'_i b$, and b is a suffix of $\boldsymbol{u}'_i$; or to $\boldsymbol{u}'_i bb$, in which case aa is a suffix of $\boldsymbol{u}'_i$. In the first case, we have $\boldsymbol{u}'_i b \leq \boldsymbol{u}'_j ab$ which implies $\boldsymbol{u}'_i \leq \boldsymbol{u}'_j$, thus $\boldsymbol{u}'_i = \boldsymbol{u}'_j$ by canonicity of the decomposition at rank $n-1$, but this is impossible since one ends with a and the other with b. In the second case, $\boldsymbol{u}'_i bb \leq \boldsymbol{u}'_j ab$, which implies $\boldsymbol{u}'_i b \leq \boldsymbol{u}'_j$, which leads to the same kind of impossibility.

In conclusion, the decomposition given in Equation 6.5 is canonical, and its number of element is given by $u_n + v_n + w_n \stackrel{\text{def}}{=} |S_{aa}| + |S_{ba}| + |S_b|$, where the sequences are inductively defined by:

$$\begin{aligned} u_{n+1} &= u_n + v_n \\ v_{n+1} &= w_n \\ w_{n+1} &= w_n + u_n \end{aligned}$$

Thus, $U_n = \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix} = A^n U_0$ where $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. This matrix has only one real eigenvalue $\lambda \simeq 1.75488 > 1$ and two conjugate complex eigenvalues $0.122561 \pm 0.744862i$ of absolute value strictly smaller than λ. As a consequence, λ^n will asymptotically dominate the expression, and the sequences u_n, v_n and w_n are exponential in n. □

A proof of Theorem 6.1.1

It is not difficult to see that atoms indeed are ideals (downward-closed and directed), and that $Idl(X^*, \leq_*)$ is closed under products. Therefore, products of atoms all are ideals, and our objective is to prove the other inclusion: all ideals of $(X^*, \leq_*)$ can be written as a finite product of atoms.

Propositions 6.1.5 and 6.1.6 do not actually rely on the structure of ideals. They can be understood as:

- The complement of a filter is a finite product of atoms.
- The finite products of atoms are closed under intersections.

Let $\boldsymbol{I} \in Idl(X^*, \leq_*)$. Its complement $\complement \boldsymbol{I}$ is upward-closed, and can therefore be decomposed $\complement \boldsymbol{I} = \bigcup_i F_i$ as a finite union of filters. It follows that $\boldsymbol{I} = \complement(\complement \boldsymbol{I}) = \bigcap_i \complement F_i$. The downward-closed sets $\complement F_i$ are finite unions of finite products of atoms (Proposition 6.1.5), and their intersection is also a finite union of finite product of atoms (Proposition 6.1.6). Finite products of atoms being ideals, $\boldsymbol{I}$ can be written as a finite union of ideals of $(X^*, \leq_*)$. But since ideals are irreducible (Proposition 2.2.1), $\boldsymbol{I}$ is a finite product of atoms.

6.1.3 Concluding Remarks

Complexity In conclusion, we have shown that all of the four main set-theoretic operations (intersection of filters and ideals, complements of filters and ideals) require exponential-time, and already for rather simple WQOs: in the most common case of sequences over a finite alphabet, already three among these four operations are exponential. Only, the quasi-ordering and ideal inclusion can be decided in polynomial time.

References and Related Work The quasi-ordering $\leq_*$ was studied by Higman in [1] where he proved that $(X^*, \leq_*)$ is a WQO if and only if $(X, \leq)$ is.

The structure of the ideals of $(X^*, \leq_*)$ was first studied by Jullien in [33], in the case where X is a finite alphabet. His proof is essentially the one we sketched at the end of Section 6.1.2. His results were later generalized to any WQO X by Kabil and Pouzet in [12]. They present a different proof, and further generalizations of this result to quasi-orderings that are out of scope of this thesis.

Abdulla, Bouajjani and Jonsson independently implicitly rediscovered the structure of the ideals of $(A^*, \leq_*)$, where A is a finite alphabet, in [17]. They use the fact, due to Higman, that downward-closed languages are regular, and prove by induction over regular expressions that for every regular expression recognizing a downward-closed language, there exists a *simple regular expression (SRE)* recognizing the same language, where simple regular expressions are exactly finite unions of finite products of atoms.

They also provide a linear-time algorithm to decide inclusion between downward-closed sets represented as SREs, and they define the same normal form as ours for SREs, which is computable in quadratic-time from any other representation. In [19], Abdulla et al. prove that downward-closed sets can be represented as SREs using the same proof as the one Jullien presented in [33], *i.e.* which we generalized for any WQO X at the end of Section 6.1.2. In particular, this proof requires to express $\complement\uparrow \boldsymbol{w}$ as an SRE and to show that SREs are closed under intersection. In other terms, they essentially prove (CF) and (II).

In [18], Abdulla et al. use the WQO $((A^{\circledast})^*, \leq_*)$, where A is a finite alphabet, and finite multisets over A (see Section 7.1) are quasi-ordered with $\leq_{\text{emb}}$. They provide a structure to represent downward-closed sets of $(A^{\circledast}, \leq_{\text{emb}})$ and $((A^{\circledast})^*, \leq_*)$ which is essentially the one obtained when composing results from this chapter with Section 7.1.

Here also, these results have been generalized to Noetherian spaces in [13]. The structure of the ideals then is still the same, and their proof is similar to ours.

Finally, we would like to point out that some alternative representations of downward-closed sets are investigated, in particular in the case of finite sequences over a finite alphabet, since it is the most common case (with motivations coming from language theory). For instance in [34], the authors represent closed sets with automata, and study the state complexity of closure operations on regular languages (represented as deterministic, non deterministic or alternating automata).

6.2 Finite Sequences under Stuttering

Let $(X, \leq)$ be a WQO. Its sequence extension under the stuttering quasi-order is $(X^*, \leq_{\text{st}})$, where X^*, as before, is the set of all finite sequences from X. Elements of X^* will be represented as in the previous section (XR). The quasi-ordering $\leq_{\text{st}}$ over X^* is defined by

$$x_1 \dots x_n \leq_{\text{st}} y_1 \dots y_m \overset{\text{def}}{\Leftrightarrow} \exists f : [n] \to [m] \text{ increasing } . \forall i \in [n].\ x_i \leq y_{f(i)}$$

The only difference with $\leq_*$ is that we do not require the witness f to be strictly increasing, but only increasing. For instance, if $X = \{a, b\}$ is a finite alphabet, then $aabbaa \leq_{\text{st}} aba \leq_{\text{st}} aabbaa$ but $aabbaa \not\leq_{\text{st}} ab$. Or with $X = \mathbb{N}$, $1 \cdot 1 \cdot 1 \leq_{\text{st}} 2$. As for $\leq_*$, $\leq_{\text{st}}$ may be decided using linear number of comparison in X, searching for the *left-most embedding* (OD). Note that even if $(X, \leq)$ is a partial-order, $(X^*, \leq_{\text{st}})$ need not be (*e.g.* $2 \cdot 2 \leq_{\text{st}} 2 \leq_{\text{st}} 2 \cdot 2$).

Another way to define this quasi-ordering is the following: define the stuttering equivalence relation $\sim_{\text{st}}$ on X^* as the smallest equivalence relation such that for all $\boldsymbol{u}, \boldsymbol{v} \in X^*$ and $a \in X$, $\boldsymbol{u}a\boldsymbol{v} \sim_{\text{st}} \boldsymbol{u}aa\boldsymbol{v}$. Informally, this equivalence does not distinguish between words which differ only in the number of times consecutive characters are repeated. Then, $\leq_{\text{st}} = \leq_* \circ \sim_{\text{st}}$, where $\circ$ denotes the composition of relations, as defined in Section 4.2. However, the results from Section 4.2 cannot be applied to this quasi-ordering since $\leq_{\text{st}} \neq \sim_{\text{st}} \circ \leq_*$. If X is a finite alphabet, this equation holds and $(X^*, \leq_{\text{st}})$ can be treated as a quotient. Also observe that $\sim_{\text{st}}$ is *not* the same as the equivalence relation $\equiv_{\text{st}} = \leq_{\text{st}} \cap \geq_{\text{st}}$ induced by the quasi-ordering, even if $(X, \leq)$ is a partial-order $\leq$. For instance, if $a \leq b$ in X, then $ab \equiv_{\text{st}} b$ in X^*, but $ab \sim_{\text{st}} b$ does not hold. However the inclusion $\sim_{\text{st}} \subseteq \equiv_{\text{st}}$ is always valid.

6.2.1 The Ideals of $(X^*, \leq_{\mathrm{st}})$

Obviously, $\leq_{\mathrm{st}}$ is an extension of $\leq_*$, thus $\leq_{\mathrm{st}}$ is a WQO and Section 4.1 applies. That is, the ideals of $(X^*, \leq_{\mathrm{st}})$ are of the form $\downarrow_{\leq_{\mathrm{st}}} I$ for I an ideal of $(X^*, \leq_*)$. Moreover,

- if D is a downward-closed subset of X, then $\downarrow_{\leq_{\mathrm{st}}} D^* = D^*$,
- if I is an ideal of $(X, \leq)$, then $\downarrow_{\leq_{\mathrm{st}}}(I + \epsilon) = I^*$,
- if $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are ideals of $(X^*, \leq_*)$ (*cf.* Section 6.1), then $\downarrow_{\leq_{\mathrm{st}}}(\boldsymbol{P}_1 \cdot \boldsymbol{P}_2) = (\downarrow_{\leq_{\mathrm{st}}} \boldsymbol{P}_1) \cdot (\downarrow_{\leq_{\mathrm{st}}} \boldsymbol{P}_2)$.

Therefore, the downward closure for $\leq_{\mathrm{st}}$ of an ideal of $(X^*, \leq_*)$ can be written as a product of atoms of the form D^*, the atoms of the form $(I + \epsilon)$ being transformed into I^*.

Lemma 6.2.1. (IR)*: Ideals of* $(X^*, \leq_{\mathrm{st}})$ *are products* $D_1^* \cdots D_n^*$*, for* $D_i \in \mathit{Down}(X)$.

6.2.2 The Stuttering Quasi-Ordering is Ideally Effective

Regarding ideal effectiveness, we show that the functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ introduced in Section 4.1 are computable, which proves that $(X^*, \leq_{\mathrm{st}})$ is ideal effective when $(X, \leq)$ is.

The function $\mathcal{C}l_{\mathrm{I}}$ is easily shown computable using the equations above: given a product of atoms $\boldsymbol{P} = \boldsymbol{A}_1 \cdots \boldsymbol{A}_n$, $\mathcal{C}l_{\mathrm{I}}(\boldsymbol{P}) = \downarrow_{\leq_{\mathrm{st}}} \boldsymbol{P} = (\downarrow_{\leq_{\mathrm{st}}} \boldsymbol{A}_1) \cdots (\downarrow_{\leq_{\mathrm{st}}} \boldsymbol{A}_n)$ and the image of an atom by $\mathcal{C}l_{\mathrm{I}}$ is obtained by $\mathcal{C}l_{\mathrm{I}}(D^*) = \downarrow_{\leq_{\mathrm{st}}} D^* = D^*$ and $\mathcal{C}l_{\mathrm{I}}(I+\epsilon) = I^*$. Thus $\mathcal{C}l_{\mathrm{I}}$ is computable in linear-time.

Function $\mathcal{C}l_{\mathrm{F}}$ is computable as well, although less straight-forward:

Proposition 6.2.2. *Given* $\boldsymbol{u} = x_1 \cdots x_n \in X^*$*,*

$$\mathcal{C}l_{\mathrm{F}}(\boldsymbol{u}) \;=\; \uparrow_{\mathrm{st}} \boldsymbol{u} \;=\; \uparrow_* \left\{ y_1 \cdots y_k \mid \begin{array}{c} 0 < k \leq n \\ 0 = i_0 < i_1 < \cdots < i_k = n \\ \forall j \in [k].\ y_j \in \min(\bigcap_{i_{j-1} < \ell \leq i_j} \uparrow_X x_\ell) \end{array} \right\}$$

Intuitively, the set ranges over all ways to cut $\boldsymbol{u}$ *in* k *pieces (factor), and embeds all elements of the* i*-th piece into the same element* y_i*.*

This is the fully generic formula to describe function $\mathcal{C}l_{\mathrm{F}}$ for any X. Since there are exponentially many family of indexes i_j to range over, this expression is computable in exponential-time. In Section 6.2.3, we show that this blow-up is unavoidable, already for a WQO as simple as $\mathbb{N}^2$. However, in simple cases, $\mathcal{C}l_{\mathrm{F}}(\boldsymbol{w})$ takes a much simpler form and is computable in linear-time. For instance, for $X = \mathbb{N}$, $\mathcal{C}l_{\mathrm{F}}(x_1 \cdots x_n) = \uparrow_*(\max_{1 \leq i \leq n} x_i)$, and for $X = \Sigma$ a finite alphabet, $\mathcal{C}l_{\mathrm{F}}(\boldsymbol{w}) = \uparrow_* \boldsymbol{v}$ where $\boldsymbol{v}$ is the shortest (in length) element of the class of $\boldsymbol{w}$ for $\sim_{st}$ (that is, $\boldsymbol{v}$ is the word $\boldsymbol{w}$ where we remove all stuttering).

Proof. ($\subseteq$) Given $\boldsymbol{w} \geq_{\mathrm{st}} x_1 \cdots x_n$, there exist an increasing mapping p from $[n]$ to $[|\boldsymbol{w}|]$ such that each x_i is associated to a greater element in $\boldsymbol{w}$. Denoting the image of p by $\{y_1, \ldots, y_k\}$, this entails a decomposition of $\boldsymbol{w} = \boldsymbol{w}_0 y_1 \boldsymbol{w}_1 y_2 \cdots y_k \boldsymbol{w}_k$ where the y_i's are in X and the $\boldsymbol{w}_i$'s in X^*. Further define i_j to be the greatest i such that $p(i) = j$ (*i.e.* the index of the right-most symbol of $x_1 \cdots x_n$ to be mapped to y_j). It follows that $0 = i_0 < i_1 < \cdots < i_k = n$, and for all $\ell \in [n]$ and $j \in [k]$, $i_{j-1} < \ell \leq i_j \Rightarrow x_\ell \leq y_j$. Then $\boldsymbol{w} \geq_* y_1 \cdots y_k$ which is indeed an element of the set described in the proposition.

The other inclusion is obvious. ☐

Applying the results of Section 4.1, we have established that $(X^*, \leq_{\mathrm{st}})$ is an ideally effective WQO. However, procedures presented in Section 4.1 rely on functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ on the one hand, and on procedures for $(X^*, \leq_*)$ on the other hand, and will thus inherit their complexity. As a consequence, and since $\mathcal{C}l_{\mathrm{I}}$ is computable in linear-time, our procedure to intersect and complement ideals of $(X^*, \leq_{\mathrm{st}})$ run in exponential-time (inherited from $(X^*, \leq_{\mathrm{st}})$), and this is shown unavoidable in Proposition 6.2.4. For procedures on filters, the complexity upper bound is less obvious, since both $\mathcal{C}l_{\mathrm{F}}$ and the procedures from $(X^*, \leq_*)$ are exponential. However, the conditions from Proposition 4.1.3 are fulfilled, from which the exponential upper bound follows. A matching lower bound for intersection of filters (IF) is proved in Proposition 6.2.4. Such a lower bound cannot be proved for (CF), since there exists a procedure to complement filters of $(X^*, \leq_{\mathrm{st}})$ that is both simpler and more efficient (linear-time computable):

Proposition 6.2.3. (CF)*:* $X^* \setminus \uparrow_{\mathrm{st}}(x_1 \cdots x_n) = (X \setminus \uparrow x_1)^* \cdots (X \setminus \uparrow x_n)^*$

Proof. ($\subseteq$) Let $y_1 \cdots y_m$ that is not greater than $x_1 \cdots x_n$ for $\leq_{\mathrm{st}}$. Consider $f : [k] \to [m]$ (for some $k < n$) the longest left-most embedding of $x_1 \cdots x_n$ into $y_1 \cdots y_m$, that is $x_1 \cdots x_k \leq_{\mathrm{st}} y_1 \cdots y_m$ but $x_1 \cdots x_k \cdot x_{k+1} \not\leq_{\mathrm{st}} y_1 \cdots y_m$. Since this is the left-most embedding, the elements y_i for $i < f(1)$ are not in $\uparrow_X x_1$, and thus $y_1 \cdots y_{f(1)-1} \in (\complement \uparrow_X x_1)^*$. Similarly, $y_{f(1)} \cdots y_{f(2)-1} \in (\complement \uparrow_X x_2)^*$ (consider this sequence to be empty if $f(1) = f(2)$). And so on, up to $y_{f(k)} \cdots y_m \in (\complement \uparrow_X x_{k+1})^*$, since otherwise we would have $x_1 \cdots x_k \cdot x_{k+1} \leq_{\mathrm{st}} y_1 \cdots y_m$. In the end we have shown that $y_1 \cdots y_m \in (\complement \uparrow_X x_1)^* \cdots (\complement \uparrow_X x_{k+1})^* \subseteq (\complement \uparrow_X x_1)^* \cdots (\complement \uparrow_X x_n)^*$.

($\supseteq$) Let $\boldsymbol{v} = \boldsymbol{v}_1 \cdots \boldsymbol{v}_n \in (\complement \uparrow_X x_1)^* \cdots (\complement \uparrow_X x_n)^*$, where $\boldsymbol{v}_i \in (\complement \uparrow_X x_i)^*$ for $i \in [n]$. Assume $x_1 \cdots x_n \leq_{\mathrm{st}} \boldsymbol{v}$, consider an embedding $f : [n] \to [|v|]$ that witnesses this inequality and consider the function g that maps $i \in [n]$ to $j \in [n]$ if the $f(i)$-th element of the sequence $\boldsymbol{v}$ is in $\boldsymbol{v}_j$. Then, because $\boldsymbol{v}_1 \in (\complement \uparrow_X x_1)^*$, $g(1) > 1$. Moreover, $g(2) \geq g(1) > 1$ but $v_2 \in (\complement \uparrow_X x_2)^*$ thus $g(2) > 2$. And so on by induction we show that $g(i) > i$, which is impossible for $g(n)$, reaching a contradiction and proving that $x_1 \cdots x_n \not\leq_{\mathrm{st}} \boldsymbol{v}$. ☐

6.2.3 Complexity Lower Bounds

Function $\mathcal{C}l_{\mathrm{F}}$

In the next proposition, we show that the upward-closed set $\uparrow_{\mathrm{st}} w$ might have exponentially many minimal elements for $\leq_*$.

Proposition 6.2.4. *Let $\boldsymbol{w}_n = \langle 0, n\rangle\langle 1, n-1\rangle \cdots \langle n, 0\rangle \in (\mathbb{N}^2)^*$ (it consists of all elements of $\mathbb{N}^2$ whose sum is equal to n). The set $\uparrow_{\mathrm{st}} \boldsymbol{w}_n$, where $\mathbb{N}^2$ is equipped with the product ordering, has exponentially many minimal elements for $\leq_*$. In particular, function Cl_{F} requires exponential-time to compute.*

Proof. Applying Proposition 6.2.2:

$$
\begin{aligned}
\uparrow_{\mathrm{st}} w_n = \quad & \uparrow_* \langle n, n\rangle & \cup \\
\bigcup_{i=0}^{n-1} \quad & \uparrow_* \langle i, n\rangle\langle n, n-i-1\rangle & \cup \\
\bigcup_{0\leq i<j<n} \quad & \uparrow_* \langle i, n-1\rangle\langle j, n-i-1\rangle\langle n, n-j-1\rangle & \cup \\
& \ldots \\
\bigcup_{\substack{0\leq i_1<\ldots \\ <i_{k-1}<n}} \quad & \uparrow_* \langle i_1, n-1\rangle\langle i_2, n-i_1-1\rangle \cdots (n, n-i_{k-1}-1\rangle & \cup \\
& \ldots \\
& \uparrow_* \boldsymbol{w}_n
\end{aligned}
$$

Each line in the above description corresponds to a value of $k \in [n]$: the first line corresponds to $k = 1$, $\boldsymbol{w}_n$ is decomposed in one piece, the second line to $k = 2$, $\boldsymbol{w}_n$ is decomposed in two pieces, etc., and the last line is obtained for $k = n$. We now argue that this decomposition of the upward closed set $\uparrow_{\mathrm{st}} \boldsymbol{w}_n$ is canonical, *i.e.* each sequence is minimal (for $\leq_*$) in $\uparrow_{\mathrm{st}} \boldsymbol{w}_n$. Indeed, let $0 < k \leq n$ and $i_0 = 0 < i_1 < \cdots < i_{k-1} < i_k = n$, and $\boldsymbol{u} = \Pi_{j=1}^{k}\langle i_j, n-i_{j-1}-1\rangle = \langle i_1, n-1\rangle\langle i_2, n-i_1-1\rangle \cdots \langle n, n-i_{k-1}-1\rangle$. We show that if $\boldsymbol{v} <_* \boldsymbol{u}$ then $\boldsymbol{w}_n \not\leq_{\mathrm{st}} \boldsymbol{v}$. For any such $\boldsymbol{v} <_* \boldsymbol{u}$, there exists an index $1 \leq \ell \leq k$ such that $\boldsymbol{v}$ is smaller than some word obtained from $\boldsymbol{u}$ by replacing the ℓ-th symbol by a strictly smaller element. That is, $\boldsymbol{v}$ is smaller than some $\langle i_1, n-1\rangle\langle i_2, n-i_1-1\rangle \cdots \langle i_{\ell-1}, n-i_{\ell-2}-1\rangle \cdot x \cdot \langle i_{\ell+1}, n-i_\ell-1\rangle \cdots \langle n, n-i_{k-1}-1\rangle$ for some $1 \leq \ell < k$ and $x <_\times \langle i_\ell, n-i_{\ell-1}\rangle$. But then, $\boldsymbol{w}_n \not\leq_{\mathrm{st}} \boldsymbol{v}$ since letter $\langle i_\ell, n-i_\ell\rangle$ isn't smaller than any letter in $\boldsymbol{v}$. □

Set-theoretic operations of $(X^*, \leq_{\mathrm{st}})$

Previously, we have exhibited a linear-time procedure for (CF). We show below that for the three remaining operations, (II), (CI) and (IF), no generic polynomial-time procedure exist.

Our first observation is that, unlike in the case of $(X^*, \leq_*)$, we won't be able to provide an exponential-lower bound in the simple case of a two-symbol alphabet. Indeed, when quotiented by $\sim_{\mathrm{st}}$, the structure of $\{a, b\}^*$ is very simple: there are only two words of any given size $n > 0$: $(ab)^(n/2)$ and $(ba)^(n/2)$ if n is even, $(ab)^m a$ and $(ba)^m b$ if $n = 2 * m + 1$.

. As a result, $(\{a, b\}^*, \leq_{\mathrm{st}})$ is isomorphic to $\mathbf{0} \oplus (\mathbb{N} \times \{a, b\}, \leq_{\mathrm{lex}})$. All operations are computable in polynomial-time in this WQO. We thus consider $A =$

$\{a, b, c\}$ a three-symbol alphabet, and prove that (II), (CI) and (IF) for $(A^*, \leq_{\mathrm{st}})$ require exponential-time computations.

(II): The canonical ideal decomposition of the downward-closed set $(a^*b^*c^*)^n \cap (b^*a^*c^*)^n$ has exponential size.

Indeed, all ideals of the form $x_1^* c^* x_2^* \cdots x_n^* c^*$ for $x_i \in \{a, b\}$ are maximal for inclusion in $(a^*b^*c^*)^n \cap (b^*a^*c^*)^n$.

(CI): The upward-closed set $\complement(a^*(b+c)^*)^n$ has exponentially many minimal elements.

Indeed, we prove that every word $x_1 a x_2 \cdots x_n a$ for $x_i \in \{b, c\}$ is a minimal element of this upward-closed set.

Let $\boldsymbol{u} = x_1 a x_2 \cdots x_n a$ be such a word. Observe that $\boldsymbol{u} \in (a^*(b+c)^*)^n$ if and only if $I \stackrel{\text{def}}{=} x_1^* a^* \cdots x_n^* a^* \subseteq (a^*(b+c)^*)^n$. This cannot be the case: there are n occurences of a^* on both sides, hence the a^* must be mapped to each other. That leaves no option for x_1^*. Therefore, $\boldsymbol{u} \in \complement(a^*b^*c^*)^n$.

It remains to show that $\boldsymbol{u}$ is minimal. If $\boldsymbol{v}$ is strictly smaller (for stuttering) than $\boldsymbol{u}$, then it belongs to $I_i = x_1^* a^* \cdots a^* x_i^* x_{i+1}^* a^* \cdots x_n^* a^*$ for some $i \in [n]$, or to $J_i = x_1^* a^* \cdots x_{i-1}^* a^* a^* x_{i+1}^* \cdots x_n^* a^*$ for some $i \in [n]$. The ideal I_i is I where we removed an atom a^* while J_i is I where we removed the atom x_i^*. Now, because $a^* a^* = a^*$ and $x_i^* x_{i+1}^* \subseteq (b+c)^*$, all ideals I_i and J_i are subsets of $(a^*(b+c)^*)^n$, which proves the minimality of $\boldsymbol{u}$.

(IF): The upward-closed set $\uparrow_{\mathrm{st}}(ac)^n \cap \uparrow_{\mathrm{st}}(bc)^n$ has exponentially many minimal elements. Indeed, applying Section 4.1:

$$\begin{aligned}\uparrow_{\mathrm{st}}(ac)^n \cap \uparrow_{\mathrm{st}}(bc)^n &= \mathcal{C}l_{\mathrm{F}}(\uparrow_{\mathrm{st}}(ac)^n) \cap \mathcal{C}l_{\mathrm{F}}(\uparrow_{\mathrm{st}}(bc)^n) \\ &= \uparrow_*(ac)^n \cap \uparrow_*(bc)^n\end{aligned}$$

This last set has exponentially many minimal elements (for $\leq_*$): at least one per element in $a^n \sqcup b^n$. Moreover, since there is no repetition of a letter in either $(ac)^n$ or $(bc)^n$, none of these minimal elements have repetitions either (*i.e.* none can be written $\boldsymbol{u} = \boldsymbol{u}_1 aa \boldsymbol{u}_2$ or $\boldsymbol{u} = \boldsymbol{u}_1 bb \boldsymbol{u}_2$). Thus, on these words, $\leq_*$ and $\leq_{\mathrm{st}}$ coincide, and these exponentially many minimal elements are also minimal for $\leq_{\mathrm{st}}$.

6.3 Finite Sequences on a Circle

Consider a WQO $(X, \leq)$, and define an equivalence relation $\sim_{\mathrm{cj}}$ on X^* as follows: $\boldsymbol{u} \sim_{\mathrm{cj}} \boldsymbol{v}$ iff there exist $\boldsymbol{w}, \boldsymbol{t}$ such that $\boldsymbol{u} = \boldsymbol{w}\boldsymbol{t}$ and $\boldsymbol{v} = \boldsymbol{t}\boldsymbol{w}$. One can imagine an equivalence class of $\sim_{\mathrm{cj}}$ as a word written on an (oriented) circle instead of a line. We can now define a notion of *subwords under conjugacy* via $\leq_{\mathrm{cj}} \stackrel{\text{def}}{=} \sim_{\mathrm{cj}} \circ \leq_*$, which is exactly the relation denoted $\preceq_c$ in [19, p.49]. The quasi-ordering $x \leq_{\mathrm{cj}} y$ can be tested using $|x|$ times a procedure for $\leq_*$, therefore in polynomial-time (OD). Elements of X^* are again encoded as before (XR).

6.3.1 The Conjugacy Quasi-Ordering is Ideally Effective

Since $\leq_* \circ \sim_{\mathrm{cj}} = \sim_{\mathrm{cj}} \circ \leq_*$, the results from Section 4.2 apply to $(X^*, \leq_{\mathrm{cj}})$. Subsequently, we prove that the functions $\mathcal{C}l_{\mathrm{F}}$ and $\mathcal{C}l_{\mathrm{I}}$ are computable, proving that $(X^*, \leq_{\mathrm{cj}})$ is ideally effective.

Recall from Section 4.2 that in this context, $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ simply corresponds to closure under $\sim_{\mathrm{cj}}$. Notably, $\mathcal{C}l_{\mathrm{F}}(\uparrow_* \boldsymbol{w}) = \uparrow_{\mathrm{cj}} \boldsymbol{w} = \uparrow_* [\boldsymbol{w}]_{\sim_{\mathrm{cj}}}$. Here, the equivalence class of some $\boldsymbol{w} \in X^*$ is given by $\{c^{(i)}(\boldsymbol{w}) \mid 1 \leq i \leq |\boldsymbol{w}|\}$, where $c^{(i)}$ designates the i-th iterate of the cycle operator $c(w_1 \cdots w_n) = w_2 \cdots w_n w_1$, which corresponds to rotating the sequence i times. Thus, function $\mathcal{C}l_{\mathrm{F}}$ is computable in linear-time.

Function $\mathcal{C}l_{\mathrm{I}}$ is similar: remember ideals of $(X^*, \leq_*)$ are finite sequences of atoms, where atoms are either D^* for some downward-closed set D of X, or $I + \epsilon$, for I some ideals of X. Then, given $\boldsymbol{P} = \boldsymbol{A}_1 \cdots \boldsymbol{A}_k$ an ideal of $(X^*, \leq_*)$:

$$\mathcal{C}l_{\mathrm{I}}(\boldsymbol{P}) = \overline{\boldsymbol{P}} = \bigcup_{i=1}^{k} c^{(i)}(\boldsymbol{P}) \cdot e(\boldsymbol{A}_i)$$

where $e(D^*) = D^*$ and $e(I + \epsilon) = \epsilon$. The presence of the extra $e(\boldsymbol{A}_i)$ in the above expression might become clearer when considering a simple example as $\boldsymbol{P} = a^* b^*$. Indeed, $aabb \in \boldsymbol{P}$, thus $abba \in \mathcal{C}l_{\mathrm{I}}(\boldsymbol{P})$.

Since the expressions above are computable in polynomial-time, procedures to intersect and complement filters and ideals of $(X^*, \leq_{\mathrm{cj}})$ immediately inherit the exponential-time upper-bound of procedures for $(X^*, \leq_*)$. In Section 6.3.2 we briefly show that all lower bounds proved for $(X^*, \leq_*)$ also hold for $(X^*, \leq_{\mathrm{cj}})$.

6.3.2 Complexity Lower Bounds

Exponential lower bounds for $(X^*, \leq_{\mathrm{cj}})$ are obtained for the same families that where used for $(X^*, \leq_*)$.

(CF): Filters (for $\leq_*$) of sequences over a finite alphabet can be complemented in polynomial time, hence, so can filters for $\leq_{\mathrm{cj}}$ ($\mathcal{C}l_{\mathrm{F}}$ is computable in polynomial-time). Consider, as in the case of the Higman quasi-ordering, the finite ordering $(X, \leq)$ consisting of three element $0, 1$ and $1'$ such that $0 \leq 1, 1'$. Since $\uparrow_{\mathrm{cj}} 0^{n+1} = \uparrow_* 0^{n+1}$, $X^* \smallsetminus \uparrow_{\mathrm{cj}} 0^{n+1} = \bigcup_{\boldsymbol{u} \in (1+1')^n} \uparrow_{\mathrm{cj}} \boldsymbol{u}$. Moreover, given two sequences $\boldsymbol{u}, \boldsymbol{v} \in X^*$ of the same length, $\boldsymbol{u} \leq_{\mathrm{cj}} \boldsymbol{v}$ iff $\boldsymbol{u} \sim_{\mathrm{cj}} \boldsymbol{v}$. Thus, the canonical ideal decomposition of $X^* \smallsetminus 0^{n+1}$ is obtained by removing equivalent elements in the expression above. However, there are at most n sequences in the equivalence class of some sequence $\boldsymbol{u}$ of length n. Thus, the canonical ideal decomposition of the downward-closed set above has at least $2^n/n$ maximal elements.

(IF): In a similar manner, since $\uparrow_{\mathrm{cj}} a^n = \uparrow_* a^n$ and $\uparrow_{\mathrm{cj}} b^n = \uparrow_* b^n$, the canonical filter decomposition of $\uparrow_{\mathrm{cj}} a^n \cap \uparrow_{\mathrm{cj}} b^n$ has exponentially many minimal elements.

(CI): Here, we again consider the same example as in Section 6.1: complementing the family of ideals $\downarrow_{\mathrm{cj}} (ab)^n$.

Let $m \in \mathbb{N}$ and $n = 5m - 1$. Subsequently, we exhibit a family $\mathcal{F}$ elements of $\downarrow_{\mathrm{cj}}(ab)^n$ such that at least exponentially many (in m, and thus in n) members of $\mathcal{F}$ are minimal in $\downarrow_{\mathrm{cj}}(ab)^n$.

Define the family $\mathcal{F}$ as the family of sequences $\boldsymbol{u} = a^{k_1} b^{k'_1} a^{k_2} \cdots a^{k_m} b^{k'_m}$ such that for all i, $k_i \geq 2$, $k'_i \geq 2$ and $\sum_{i=1}^{m} k_i + k'_i = |u| = n + m + 1$.

Elements of $\mathcal{F}$ are in $X^* \smallsetminus \downarrow_{\mathrm{cj}}(ab)^n$. Indeed, for $p, q \in \mathbb{N}$, $a^p b^q \leq_{\mathrm{cj}} (ab)^{p+q-1}$ but $a^p b^q \not\leq_{\mathrm{cj}} (ab)^{p+q-2}$. Thus, for $\boldsymbol{u} \in \mathcal{F}$, $\boldsymbol{u} \leq_{\mathrm{cj}} (ab)^{\sum_{i=1}^{m} k_i + k'_i - m} = (ab)^{n+1}$ but $\boldsymbol{u} \not\leq_{\mathrm{cj}} (ab)^n$. It follows that all elements from $\mathcal{F}$ are minimal for $\leq_{\mathrm{cj}}$ in $X^* \smallsetminus \downarrow_{\mathrm{cj}}(ab)^n$ (removing a symbol in $\boldsymbol{u} \in \mathcal{F}$ would entail $\boldsymbol{u} \leq_{\mathrm{cj}} (ab)^n$).

Moreover, members of $\mathcal{F}$ have the same length, hence they are either equivalent under $\sim_{\mathrm{cj}}$ or incomparable with respect to $\leq_{\mathrm{cj}}$. Since equivalence classes are of linear size in the length of the sequence, and the length of the sequences described above are linear in n, it suffices to show that $\mathcal{F}$ has exponentially many elements to conclude.

Observe that we can assign either $(2,3)$ or $(3,2)$ to the first $m-1$ pairs (k_i, k'_i), and use the last pair to make the sum equal to $n+1+m$. Indeed, $\sum_{i=1}^{m} k_i + k'_i = 5(m-1) + k_m + k'_m = n - 4 + k_m + k'_m$. Therefore, it suffices to chose $k_m + k'_m = 4 + m + 1$, which is always possible. This proves that $\mathcal{F}$ has at least 2^{m-1} elements, among which at least $\frac{2^{m-1}}{n+m+1}$ are minimal in $\downarrow_{\mathrm{cj}}(ab)^n$.

(II): Again, the example from Section 6.1 still witnesses an exponential blow-up. We have:

$$\begin{aligned} D &= \downarrow_{\mathrm{cj}}(aba)^n \cap \downarrow_{\mathrm{cj}}(bab)^n \\ &= \downarrow_*((aba)^n \cup (baa)^n \cup (aab)^n) \cap \downarrow_*((bab)^n \cup (abb)^n \cup (bba)^n) \end{aligned}$$

We know the decomposition of $(aba)^n \cap (bab)^n$ already contains exponentially many maximal sequences (see Section 6.1). These sequences are still maximal in D, since all sequences in D have length bounded by $2n$. Moreover, since all these maximal sequences have same length $2n$, the family remains exponential when quotiented by $\sim_{\mathrm{cj}}$.

Chapter 7

Finite Multisets of WQOs

Given a WQO $(X, \leq)$, we consider the set $X^{\circledast}$ of finite multisets over X. Intuitively, multisets are sets where an element might occur multiple times. Formally, a multiset $M \in X^{\circledast}$ is a function from X to $\mathbb{N}$: $M(x)$ denotes the number of occurrences of x in M. The support of a multiset M denoted $Supp(M)$ is the set $\{x \in X \mid M(x) \neq 0\}$. A multiset is said to be finite if its support is.

A natural algorithmic representation for these objects are lists of elements of X, but keeping in mind that a permutation of a list represents the same multiset. Formally, this means that $X^{\circledast}$ is the quotient of X^* by the equivalence relation $\sim$ defined for all $\boldsymbol{u}, \boldsymbol{v} \in X^*$ by $\boldsymbol{u} \sim \boldsymbol{v}$ iff the sequence $\boldsymbol{u}$ can be obtained by permuting the symbols in $\boldsymbol{v}$, *i.e.* formally if $\boldsymbol{u} = u_1 \cdots u_n$ and $\boldsymbol{v} = v_1 \cdots v_m$,

$$\boldsymbol{u} \sim \boldsymbol{v} \overset{\text{def}}{\Leftrightarrow} n = m \wedge \exists \sigma \in S_n.\, \forall i \in [n].\, u_i = v_{\sigma(i)}$$

where S_n denotes the permutation group over $[1, n]$.

Therefore, finite multisets are equivalence classes of sequences, and we represent them by any member of the class, between special brackets. For instance, if $X = \{a, b, c\}$ is a finite alphabet, $M = \{\!|aac|\!\}$ is the multiset such that $M(a) = 2$, $M(c) = 1$ and $M(b) = 0$. Its other representations are $M = \{\!|aca|\!\} = \{\!|caa|\!\}$.

Sometimes, it will be more convenient to use the functional point of view, and we will define some multiset M by providing values $M(x)$ for every $x \in X$. It is necessary to check that multisets defined in this fashion are finite.

Below we introduce notations for several natural operations, in particular generalizations of set-theoretic operations:

- Multiset union: $(M_1 + M_2)(x) = M_1(x) + M_2(x)$. It is clear from that definition why we denote this operation with the symbol $+$, but keep in mind that this operation simply is the quotiented version of sequence concatenation, that is $\{\!|\boldsymbol{uv}|\!\} = \{\!|\boldsymbol{u} \cdot \boldsymbol{v}|\!\} = \{\!|\boldsymbol{u}|\!\} + \{\!|\boldsymbol{v}|\!\}$. As in the case of sequences, we generalize this operation to set of multisets. If $\boldsymbol{U}, \boldsymbol{V}$ are sets of sequences, $\boldsymbol{U} \cdot \boldsymbol{V} = \{\boldsymbol{u} \cdot \boldsymbol{v} \mid \boldsymbol{u} \in \boldsymbol{U}, \boldsymbol{v} \in \boldsymbol{V}\}$. Similarly, given $\boldsymbol{S}, \boldsymbol{T}$ sets of multisets, we note $\boldsymbol{S} \oplus \boldsymbol{T} = \{M + N \mid M \in \boldsymbol{S}, N \in \boldsymbol{T}\}$. In particular, $\overline{\boldsymbol{U} \cdot \boldsymbol{V}} = \overline{\boldsymbol{U}} \oplus \overline{\boldsymbol{V}}$. We chose the notation $\oplus$

instead of overloading the symbol $+$ in order to avoid possible confusion with set union.

- Multiset difference: $(M_1 - M_2)(x) = \max(0, M_1(x) - M_2(x))$.
- Membership: $x \in M \Leftrightarrow x \in Supp(M) \Leftrightarrow M(x) > 0$.
- Inclusion: $M_1 \subseteq M_2 \Leftrightarrow \forall x \in X.\ M_1(x) \leq M_2(x)$.
- Set intersection, or infimum: $(M_1 \cap M_2)(x) = \min(M_1(x), M_2(x))$.
- Set union, or supremum: $(M_1 \cup M_2)(x) = \max(M_1(x), M_2(x))$.
- Cardinality: $|M| = \sum_{x \in X} M(x)$, or if $w \in X^*$ is a member of the equivalence class M, $|M| = |\{\!|w|\!\}| = |w|$. We will thus often refer to $|M|$ as the *length* of M, or its *size*.
- Restriction: given $Y \subseteq X$, the restriction of M seen as a function to the domain Y is defined by:

$$M_{|Y}(x) = \begin{cases} M(x) & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

- Set Difference: we will write $M \smallsetminus Y$ for $M_{|X \smallsetminus Y}$

We will denote the empty multiset by $\emptyset$.

7.1 Multisets under the Embedding Quasi-Ordering

The set $X^{\circledast}$ is well quasi-ordered by the embedding relation:

$$\{\!|x_1 \cdots x_n|\!\} \leq_{emb} \{\!|y_1 \cdots y_m|\!\} \overset{\text{def}}{\Leftrightarrow} \exists f : [n] \to [m] \text{ injective s.t. } \forall i \in [n], x_i \leq y_{f(i)}$$

As in the case of words, a function f that satisfies the right-hand side of the above equivalence is said to be a *witness* of $\{\!|x_1 \cdots x_n|\!\} \leq_{emb} \{\!|y_1 \cdots y_m|\!\}$.

The quasi-ordering can also be obtained as a quotient, $\leq_{\text{emb}} = \leq_* \circ \sim = \sim \circ \leq_*$, which proves that $\leq_{\text{emb}}$ is a WQO. Another convenient characterization of the quasi-ordering is obtained using Hall's Marriage Theorem.

Theorem 7.1.1 (Hall)**.** *Let $G = (V_1 \sqcup V_2, E)$ be a finite bipartite graph. E contains the graph of an injection from V_1 to V_2 if and only if for every $W \subseteq V_1$, $|W| \leq |N_G(W)|$. Here we write $N_G(W)$ for the* neighborhood *of W: $N_G(W) = \{y \mid \exists x \in W.\ (x, y) \in E\}$.*

The name of the theorem comes from the following story: interpret V_1 and V_2 as sets of men and women, and E as the relation "would like to marry each other". The theorem characterizes the cases where it is possible to happily marry the entirety of one of the two groups (but of course, if V_1 and V_2 do not have same size, some people from the most numerous group will remain single).

Proposition 7.1.2. *Let M, N be two finite multisets.*

$$N \leq_{\text{emb}} M \textit{ iff } \forall S \subseteq N.\ |S| \leq |M_{|\uparrow Supp(S)}|$$

Proof. Write $N = \{\!|x_1 \cdots x_n|\!\}$ and $M = \{\!|y_1, \dots, y_m|\!\}$. Define the bipartite graph $G = ([n] \sqcup [m], E)$ where $(i, j) \in E$ iff $x_i \leq y_j$. This is the bipartite graph with elements of N in one bag, elements of M in the other bag, and elements in N are linked to elements in M that are greater. Then

$$\begin{aligned} N \leq_{\text{emb}} M &\text{ iff there is an injection } f : [n] \to [m] \text{ such that } \forall i \in [n].\ x_i \leq y_{f(i)} \\ &\text{ iff there is an injection } f : [n] \to [m] \text{ such that } \forall i \in [n].\ (i, f(i)) \in E \\ &\text{ iff } E \text{ contains the graph of an injection} \\ &\text{ iff } \forall W \subseteq [n].\ |W| \leq |N_G(W)| \end{aligned}$$

the last equivalence being by Theorem 7.1.1. Finally, a subset W of $[n]$ defines a multiset $S \subseteq N$ and $N_G(W)$ represents exactly the sub-multiset of elements of M that are greater than some elements of S, *i.e.* $M_{|\uparrow Supp(S)}$. ☐

Corollary 7.1.3. (OD)*: The relation $\leq_{\text{emb}}$ can be decided in polynomial-time.*

Proof. In a graph $G = (V, E)$, a *matching* is a subset S of E such that all the edges of S have distinct ending points, that is no vertex is an ending points of two distinct edges in S. The injectivity condition in our statement of Hall's Theorem guarantees that the set of edges $S = \{(x, f(x)) \mid x \in V_1\}$ is a matching. The usual formulation of Hall's Theorem gives necessary and sufficient condition for the existence of a matching that covers V_1.

Given multisets N and M, consider the bipartite graph G as above. There are polynomial-time algorithms to compute the maximum size of a matching in a bipartite graph (see Ford-Fulkerson algorithm, or Hopcroft-Karp algorithm). Thus, the relation $\leq_{\text{emb}}$ can be decided in polynomial-time: it suffices to compute the maximal size of a matching in G and test whether it is equal to $|N|$. ☐

7.1.1 The Ideals of $(X^{\circledast}, \leq_{\text{emb}})$

By Proposition 4.2.1, the ideals of $(X^{\circledast}, \leq_{\text{emb}})$ are the closure under $\sim$ of the ideals of $(X^*, \leq_*)$. Remember from Section 6.1 that ideals $\boldsymbol{P} \in Idl(X^*, \leq_*)$ are products $\boldsymbol{P} = A_1 \cdots A_k$ of atoms of the form D^* for some $D \in Down(X)$ or $I + \epsilon$ for some $I \in Idl(X)$. Thus, ideals of $(X^{\circledast}, \leq_{\text{emb}})$ take the following form:

$$\begin{aligned} \overline{\boldsymbol{P}} &= \overline{A_1 \cdots A_k} \\ &= \bigoplus_{i=1}^{k} \overline{A_i} \\ &= \bigoplus_{A_i = E^*} E^{\circledast} \oplus \bigoplus_{A_i = I + \epsilon} (\{\{\!|x|\!\} \mid x \in I\} \cup \{\emptyset\}) \end{aligned}$$

Denote by $E_1, \dots, E_n$ the downward-closed sets of X and $I_1, \dots, I_m$ the ideals of X that appear in $\boldsymbol{P}$ (with multiplicity). Observe that given $D_1, D_2 \in Down(X)$,

$D_1^{\circledast} \oplus D_2^{\circledast} = (D_1 \cup D_2)^{\circledast}$. Thus, the first term simplifies to $D^{\circledast}$ for $D = \bigcup_{i=1}^{n} = E_i$, The second term consists of a set of multisets M that have at most m elements which can be injectively paired with some ideal I_i they belong to. Formally,

$$\{\!|x_1 \cdots x_p|\!\} \in \bigoplus_{i=1}^{m}(\{\{\!|x|\!\} \mid x \in I_i\} \cup \{\emptyset\}) \text{ iff}$$
$$\exists f : [p] \to [m] \text{ injective s.t.} \forall i \in [p].\ x_i \in I_{f(i)}$$

Note that this last expression is very similar to the definition $\leq_{\text{emb}}$, but with $\in$ used instead of $\leq$. This motivates the following notation:

$$\{\!|x_1 \cdots x_p|\!\} \in_{\text{emb}} \{\!|I_1 \cdots I_m|\!\} \overset{\text{def}}{\Leftrightarrow} \exists f : [p] \to [m] \text{ injective s.t.} \forall i \in [p].\ x_i \in I_{f(i)}$$

Let $\boldsymbol{B} = \{I_1 \cdots I_m\} \in \mathit{Idl}(X)^{\circledast}$ the multisets of all ideals I_i that appear in $\boldsymbol{P}$ (with multiplicity), we have proved that $\overline{P} = \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$, where $\downarrow_{\in} \boldsymbol{B} = \{M \in X^{\circledast} \mid M \in_{\text{emb}} \boldsymbol{B}\}$.

Lemma 7.1.4. (IR)*:*

$$\mathit{Idl}(X^{\circledast}, \leq_{\text{emb}}) = \{\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \mid \boldsymbol{B} \in \mathit{Idl}(X)^{\circledast}, D \in \mathit{Down}(X)\}$$

where $\downarrow_{\in} \boldsymbol{B} = \{M \in X^{\circledast} \mid M \in_{\text{emb}} \boldsymbol{B}\}$ *and* $\in_{\text{emb}}$ *is defined as* $\leq_{\text{emb}}$*, but using* $\in$ *instead of* $\leq$*.*

Before proving what remains to be proved of this lemma, we would like to present a convenient characterization of $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ that will be constantly used throughout this section. Given $\boldsymbol{B} \in \mathit{Idl}(X)^{\circledast}$ a multiset of ideals of X and $D \in \mathit{Down}(X)$ a downward-closed set of X:

$$\begin{aligned}\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} &= \{M_1 + M_2 \mid M_1 \in_{\text{emb}} \boldsymbol{B} \wedge M_2 \in D^{\circledast}\} \\ &= \{M \mid M \setminus D \in_{\text{emb}} \boldsymbol{B}\}\end{aligned}$$

Indeed, if M is such that $M \setminus D \in_{\text{emb}} \boldsymbol{B}$, then define $M_1 = M \setminus D$ and $M_2 = M_{|D}$ to satisfy $M = M_1 + M_2 \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$. For the other direction, for any decomposition $M = M_1 + M_2$ satisfying $M_1 \in_{\text{emb}} \boldsymbol{B}$ and $M_2 \in D^{\circledast}$, M_2 must be a sub-multiset of $M_{|D}$, and thus $M \setminus D \subseteq M_1$. It is thus obvious that $M \setminus D \in_{\text{emb}} \boldsymbol{B}$.

We are now ready to prove Lemma 7.1.4

Proof. ($\subseteq$): This direction follows from the investigation above.

($\supseteq$): It remains to check that all subsets of $X^{\circledast}$ of the form $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ are indeed ideals.

- Downward-closed: let $M \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$, and $N \leq_{\text{emb}} M$. Since D is downward-closed, $N \setminus D \leq_{\text{emb}} M \setminus D$ (a witness can be obtained by restricting a witness for $N \leq_{\text{emb}} M$). Moreover, composing the embeddings witnessing $N \setminus D \leq_{\text{emb}} M \setminus D$ and $M \setminus D \in_{\text{emb}} \boldsymbol{B}$, we obtain $N \setminus D \in_{\text{emb}} \boldsymbol{B}$ (elements of $\boldsymbol{B}$ are downward-closed); hence $N \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$.

- Directed: let $M, N \in {\downarrow_\in} \boldsymbol{B} \oplus D^\circledast$. Write $\boldsymbol{B} = \{\!|I_1 \cdots I_k|\!\}$. Define $P = M_{|D} + N_{|D} + \{\!|z_1 \cdots z_k|\!\}$ where for all $i \in [k]$, z_i is greater than every element of M and N that belong to I_i. Such an element z_i exists since I_i is directed, and M and N are finite. Obviously, $P \in {\downarrow_\in} \boldsymbol{B} \oplus D^\circledast$ and $M, N \leq_{\mathrm{emb}} P$. □

Observe that this representation of ideals is not unique: for instance with $X = \mathbb{N}$, ${\downarrow_\in} \{\!|{\downarrow}3 \cdot {\downarrow}1|\!\} \oplus ({\downarrow}2)^\circledast = {\downarrow_\in} \{\!|{\downarrow}3|\!\} \oplus ({\downarrow}2)^\circledast$. The question of canonical representations of ideals is addressed after we show how to decide inclusion.

Proposition 7.1.5. (ID)*: Let $\boldsymbol{B}_1, \boldsymbol{B}_2 \in \mathit{Idl}(X)^\circledast$ and $D_1, D_2 \in \mathit{Down}(X)$.*

$$ {\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast \subseteq {\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast \textit{ iff } D_1 \subseteq D_2 \textit{ and } \boldsymbol{B}_1 \smallsetminus \mathit{Down}(D_2) \subseteq_{\mathrm{emb}} \boldsymbol{B}_2 $$

This proposition implies (ID) since both inclusion of downward-closed sets of X and $\subseteq_{\mathrm{emb}}$ are decidable. Moreover, the test can be performed in polynomial-time in terms of operations in $(X, \leq)$, thanks to Corollary 7.1.3.

Proof. Write $\boldsymbol{B} = \boldsymbol{B}_1 \smallsetminus \mathit{Down}(D_2) = \{\!|I_1 \cdots I_n \cdot {\downarrow}y_1 \cdots {\downarrow}y_m|\!\}$, where $I_1, \ldots, I_n$ are limit ideals (*i.e.* those that are not principal), and $y_1, \ldots, y_m \in X$.

($\Rightarrow$) For any $x \in D_1$, the multiset with $|\boldsymbol{B}_2| + 1$ copies of x is in ${\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast$, hence it must be in ${\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast$. However $M \in_{\mathrm{emb}} \boldsymbol{B}_2$ implies $|M| \leq |\boldsymbol{B}_2|$, thus x must be in D_2, proving $D_1 \subseteq D_2$.

If $n = 0$, then $\{\!|y_1 \cdots y_m|\!\} \in {\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast \subseteq {\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast$ with $y_i \notin D_2$ by definition, entailing $\{\!|y_1 \cdots y_m|\!\} \in_{\mathrm{emb}} \boldsymbol{B}_2$. This proves $\boldsymbol{B} \subseteq_{\mathrm{emb}} \boldsymbol{B}_2$.

Otherwise, $n \neq 0$ and the set $\boldsymbol{A} = \{\{\!|x_1 \cdots x_n \cdot y_1 \cdots y_m|\!\} \mid \forall i.\ x_i \in I_i \smallsetminus D_2\}$ is infinite, included in ${\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast \subseteq {\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast$ and the only way to have $\boldsymbol{A} \subseteq {\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast$ is if $\forall M \in \boldsymbol{A}$, $M \in_{\mathrm{emb}} \boldsymbol{B}_2$. Now consider the set of embeddings from $[n+m]$ to $[|\boldsymbol{B}_2|]$, it is finite, hence infinitely many memberships are witnessed by a same embedding f. It is easy to see that f witnesses $\boldsymbol{B} \leq_{\mathrm{emb}} \boldsymbol{B}_2$.

($\Leftarrow$) Let $M \in {\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast$. Since $D_1 \subseteq D_2$, $M \smallsetminus D_2 \leq_{\mathrm{emb}} M \smallsetminus D_1$. Moreover, $M \smallsetminus D_1 \in_{\mathrm{emb}} \boldsymbol{B}_1$, and therefore $M \smallsetminus D_2 \in_{\mathrm{emb}} \boldsymbol{B}$, by definition of $\boldsymbol{B}$. We conclude $M \smallsetminus D_2 \in_{\mathrm{emb}} \boldsymbol{B}_2$. □

Definition 7.1.6. *Given $\boldsymbol{B} \in \mathit{Idl}(X)^\circledast$ and $D \in \mathit{Down}(X)$, we say that $(\boldsymbol{B}, D)$ is the* canonical representation *of the ideal ${\downarrow_\in} \boldsymbol{B} \oplus D^\circledast$ if for every $I \in \boldsymbol{B}$, $I \not\subseteq D$. This representation is unique and computable from any other representation of ${\downarrow_\in} \boldsymbol{B} \oplus D^\circledast$.*

Proof. Uniqueness: if ${\downarrow_\in} \boldsymbol{B}_1 \oplus D_1^\circledast = {\downarrow_\in} \boldsymbol{B}_2 \oplus D_2^\circledast$ and that both representations are canonical, then by Proposition 7.1.5, $D_1 = D_2$, and thus $\forall I \in \boldsymbol{B}_i, I \not\subseteq D_i$. Therefore, the second condition directly gives $\boldsymbol{B}_1 \subseteq_{\mathrm{emb}} \boldsymbol{B}_2 \subseteq_{\mathrm{emb}} \boldsymbol{B}_1$, which implies $\boldsymbol{B}_1 = \boldsymbol{B}_2$.

Computability: If $(X, \leq)$ is effective, inclusion is decidable, thus one can compute the canonical representation of an ideal from any other representation by testing for every $I \in \mathit{Supp}(\boldsymbol{B})$ whether $I \subseteq D$. If it is the case, removing all copies of I from $\boldsymbol{B}$ does not change ${\downarrow_\in} \boldsymbol{B} \oplus D^\circledast$. □

7.1.2 The Embedding Quasi-Ordering on Multisets is Ideally Effective

Since finite multisets are equivalence classes of finite sequences, one can apply Section 4.2: $(X^{\circledast}, \leq_{\text{emb}})$ is ideally effective provided that the closure functions

$$\mathcal{C}l_{\text{I}} : \mathit{Idl}(X^*, \leq_*) \to \mathit{Down}(X^*, \leq_*)$$
$$\mathcal{C}l_{\text{F}} : \mathit{Filters}(X^*, \leq_*) \to \mathit{Up}(X^*, \leq_*)$$

are computable. These two functions indeed are computable, but require exponential-time, because of all the possible permutations of a sequence (as in Section 7.3). Thus, $(X^{\circledast}, \leq_{\text{emb}})$ is ideally effective, but this approach only gives exponential-time procedures (relying on Proposition 4.1.3). Notice that we have provided a more efficient procedure to decide inclusion (Proposition 7.1.5). In Section 7.1.3, we show that for the four others operations, the exponential-time complexity is unavoidable in general. Of course, there are polynomial-time implementations in specific cases: for instance, given a finite alphabet A, $(A^{\circledast}, \leq_{\text{emb}})$ is isomorphic to $(\mathbb{N}^{|A|}, \leq_{\times})$.

Note that Section 4.2 assumes we represent ideals J of $(X^{\circledast}, \leq_{\text{emb}})$ as ideals I of $(X^*, \leq_*)$, and "remember" it stands for the closure under $\sim$, *i.e.* $J = \overline{I}$. Here we have fixed another representation for ideals. Luckily, given an ideal $\boldsymbol{P}$ of $(X^*, \leq_*)$, our representation of $\overline{\boldsymbol{P}}$ can be computed in linear-time: this conversion is implicitly described in the investigation that comes before Lemma 7.1.4. Now in order to apply the procedures from Section 4.2, we need to replace function $\mathcal{C}l_{\text{I}}$ by function $\mathcal{C}l'_{\text{I}} : \mathit{Idl}(X^{\circledast}, \leq_{\text{emb}}) \to \mathit{Down}(X^*, \leq_*)$ that converts our representation of an ideal of $(X^{\circledast}, \leq_{\text{emb}})$ into a representation of the same set of elements, but as a downward-closed set of $(X^*, \leq_*)$. Note that $\mathcal{C}l_{\text{I}}$ simply is the composition of the two conversions mentioned above.

Given $\boldsymbol{B} = \{\!|I_1 \cdots I_n|\!\} \in (\mathit{Idl}(X))^{\circledast}$ and $D \in \mathit{Down}(X)$,

$$\mathcal{C}l'_{\text{I}}(\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}) = \bigcup_{\sigma \in S_n} D^*(I_{\sigma(1)} + \epsilon)D^* \cdots D^*(I_{\sigma(n)} + \epsilon)D^*$$

where S_n denotes the group of permutations over n elements. As for $\mathcal{C}l_{\text{F}}$, Section 4.2 gives $\mathcal{C}l_{\text{F}}(w) = \uparrow_*[w]_\sim$, where $\sim$ is the equivalence relation defined at the beginning of this section. Notice that $[w]_\sim$ consists of all the possible representation of $\{\!|w|\!\}$.

These expressions prove that $(X^{\circledast}, \leq_{\text{emb}})$ is an ideally-effective WQO. However, since finite multisets are of great importance in computer science, we provide explicit expressions for our four set-theoretic operations in the next section.

7.1.3 Explicit Expressions for operations in $(X^{\circledast}, \leq_{\text{emb}})$ and Complexity

In this section, we provide direct expressions and exponential-time lower bounds for complementing filters, complementing ideals, intersecting filters and intersecting ideals, in this order. As stated before, for a finite alphabet A, $(A^{\circledast}, \leq_{\text{emb}})$ is isomorphic to $(\mathbb{N}^{|A|}, \leq_{\times})$. Thus, we have to turn to more complex WQOs to provide exponential lower bounds.

To increase readability of the proposition of this section, we will use the symbol $\times$ to denote iteration of multiset addition, *i.e.* $n \times M = \underbrace{M + \cdots + M}_{n \text{ times}}$. We also extend the definitions of $\in_{\text{emb}}$ and $\downarrow_\in$ to multisets $\boldsymbol{B} \in \mathit{Down}(X)^{\circledast}$ of downward-closed sets of X (instead of ideals). In this case, $\downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$ is not an ideal of $(X^{\circledast}, \leq_{\text{emb}})$, but it is downward-closed. Besides, if $\boldsymbol{B} = \{\!\{D_1, \ldots, D_m\}\!\}$ and for all i, $D_i = \bigcup_{j=1}^{n_i} I_{i,j}$, then

$$\downarrow_\in \boldsymbol{B} \oplus D^{\circledast} = \bigcup_{\substack{j_1 \in [n_1] \\ \cdots \\ j_m \in [n_m]}} \downarrow_\in \{\!\{I_{1,j_1} \cdots I_{m,j_m}\}\!\} \oplus D^{\circledast}$$

Thus, when a downward-closed set $\boldsymbol{D}$ of $(X^{\circledast}, \leq_{\text{emb}})$ is given as a union of such sets $\downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$, it is possible to compute the actual ideal decomposition of $\boldsymbol{D}$. Note however that the ideal decomposition of $\downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$ in this setting is polynomial in $|D_i|$ but exponential in m.

Complementing Filters (CF)

Proposition 7.1.7. *Let M be a multiset over X,*

$$X^{\circledast} \smallsetminus \uparrow M = \bigcup_{\substack{S \subseteq \mathit{Supp}(M) \\ S \neq \emptyset}} \downarrow_\in \{\!\{ \underbrace{X \cdots X}_{|M_{|S}|-1 \textit{ times}} \}\!\} \oplus [X \smallsetminus \uparrow S]^{\circledast}$$

The above expression is computable, using (XI) and (CF) for X. However, it requires exponential-time to enumerate all the subsets of M. This enumeration is shown to be unavoidable in the next proposition.

Note that X may not be an ideal, in which case the actual ideal decomposition is obtained by distributing the ideal decomposition of X as described at the beginning of this subsection.

Proof. For $T \subseteq M$, define the downward-closed set

$$\boldsymbol{D}(T) = \downarrow_\in \{\!\{ \underbrace{X \cdots X}_{|T|-1 \text{ times}} \}\!\} \oplus [X \smallsetminus \uparrow \mathit{Supp}(T)]^{\circledast}$$

$$\begin{aligned} N \in X^{\circledast} \smallsetminus \uparrow M \ &\text{iff } M \not\leq_{\text{emb}} N \\ &\text{iff } \exists T \subseteq M.\ |T| > |N_{|\uparrow \mathit{Supp}(T)}| \text{ (negating Proposition 7.1.2)} \\ &\text{iff } \bigvee_{\substack{T \subseteq M \\ T \neq \emptyset}} |N_{|\uparrow \mathit{Supp}(T)}| \leq |T| - 1 \\ &\text{iff } N \in \bigcup_{\substack{T \subseteq M \\ T \neq \emptyset}} \boldsymbol{D}(T) \end{aligned}$$

Now for $T \subseteq M$, let $S = \mathit{Supp}(T)$, it is obvious that $T \subseteq M_{|S}$ and $\boldsymbol{D}(T) \subseteq \boldsymbol{D}(M_{|S})$. We can thus restrict the union above to the union described in the proposition. $\square$

Proposition 7.1.8. *There exists an ideally effective WQO $(X, \leq)$ such that the expression provided in Proposition 7.1.7 is canonical. In particular, complementing filters of $(X^{\circledast}, \leq_{\text{emb}})$ requires exponential-time.*

Proof. Our goal is to provide a WQO X and an infinite family $(M_n)_{n\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$ and any $S, T \subseteq M_n$, $\boldsymbol{D}(S)$ and $\boldsymbol{D}(T)$ are incomparable (for inclusion), provided $S \neq T$.

One can check that this is not the case for $X = \mathbb{N}$. Indeed, as soon as M has at least two ordered elements $n_1 \leq n_2$, $\boldsymbol{D}(\{\!|n_1|\!\}) \subseteq \boldsymbol{D}(\{\!|n_1, n_2|\!\})$. Since elements of $\mathbb{N}$ are linearly-ordered, we can prove:

$$\mathbb{N}^{\circledast} \smallsetminus \uparrow M = \downarrow_{\in} \{\!|\underbrace{X \cdots X}_{|M|-1 \text{ times}}|\!\}] \oplus [\downarrow \min(M) - 1]^{\circledast}$$

We thus take $X = \mathbb{N}^2$ (which is an ideal), and define $M_n = \{\!|\langle 0, n\rangle\langle 1, n-1\rangle \cdots \langle n, 0\rangle|\!\}$. Now let S and T be two distinct subsets of M_n, we show that $\boldsymbol{D}(S) \not\subseteq \boldsymbol{D}(T)$.

- If $|S| \leq |T|$, but $S \neq T$, there exists $x \in \mathit{Supp}(T) \smallsetminus \mathit{Supp}(S)$, and since elements of M_n are pairwise incomparable, $x \notin \uparrow \mathit{Supp}(S)$. This proves that $\complement\uparrow \mathit{Supp}(S) \not\subseteq \complement\uparrow \mathit{Supp}(T)$, and by Proposition 7.1.5, $\boldsymbol{D}(S) \not\leq_{\text{emb}} \boldsymbol{D}(T)$.
- If $|S| > |T|$ then $\boldsymbol{D}(S) \not\leq_{\text{emb}} \boldsymbol{D}(T)$ since there are too many copies of X in $\boldsymbol{D}(S)$.

Notice that we have actually proved that our expression for $X^{\circledast} \smallsetminus \uparrow M$ is canonical if and only if M is an antichain. □

Complementing Ideals (CI)

The procedure to complement ideals is very similar to the procedure to complement filters:

Proposition 7.1.9. *Given $\boldsymbol{B} \in (\mathit{Idl}(X, \leq))^{\circledast}$ and $D \in \mathit{Down}(X, \leq)$,*

$$X^{\circledast} \smallsetminus (\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}) = \bigcup_{\boldsymbol{S} \subseteq \mathit{Supp}(\boldsymbol{B})} \uparrow\{M \mid \mathit{Supp}(M) \subseteq U_{\boldsymbol{S},\boldsymbol{B},D} \wedge |M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1\}$$

where $U_{\boldsymbol{S},\boldsymbol{B},D} = \min(X \smallsetminus (D \cup \bigcup(\mathit{Supp}(\boldsymbol{B}) \smallsetminus \boldsymbol{S})))$.

Concretely, minimal multisets M of $X^{\circledast} \smallsetminus \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ are multisets of size exactly $|\boldsymbol{B}_{|\boldsymbol{S}}| + 1$ whose elements are among the minimal elements of $X \smallsetminus (D \cup \bigcup(\mathit{Supp}(\boldsymbol{B}) \smallsetminus \boldsymbol{S}))$, for some $\boldsymbol{S} \subseteq \mathit{Supp}(\boldsymbol{B})$.

Proof. ($\subseteq$) Let $M \notin \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$. That is to say, $M \smallsetminus D \not\in_{\text{emb}} \boldsymbol{B}$. Adapting Proposition 7.1.2, the latter is equivalent to the existence of $T \subseteq M \smallsetminus D$ such that $|\boldsymbol{B}_{|\boldsymbol{I}(T)}| < |T|$, where $\boldsymbol{I}(T) = \{I \in \mathit{Idl}(X) \mid I \cap \mathit{Supp}(T) \neq \emptyset\}$ (understand $\boldsymbol{I}(T)$ as the set of ideals that contain some element of T). Let $\boldsymbol{S} = \mathit{Supp}(\boldsymbol{B}) \cap \boldsymbol{I}(T)$, we show that $T \in \uparrow\{M \mid \mathit{Supp}(M) \subseteq U_{\boldsymbol{S},\boldsymbol{B},D} \wedge |M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1\}$. Indeed,

$|T| \geq |\boldsymbol{B}_{|\boldsymbol{I}(T)}| + 1 = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1$. Moreover, given $x \in T$, $x \notin D$ since $T \subseteq M \smallsetminus D$, and x is not in any ideal $I \in \bigcup Supp(\boldsymbol{B}) \smallsetminus \boldsymbol{S}$ since by definition, if $x \in I$ then $I \in \boldsymbol{I}(T)$. We just showed that $Supp(T) \subseteq X \smallsetminus (D \cup \bigcup Supp(\boldsymbol{B}) \smallsetminus \boldsymbol{S})$. It is then not difficult to build a multiset $N \leq_{\text{emb}} T$ that satisfies the desired conditions.

($\supseteq$) Let M such that $Supp(M) \subseteq U_{\boldsymbol{S},\boldsymbol{B},D}$ and $|M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1\}$ for some $\boldsymbol{S} \subseteq Supp(\boldsymbol{B})$. First of all $M = M \smallsetminus D$. Besides, an injective mapping f from M to $\boldsymbol{B}$ would define a submultiset of $\boldsymbol{B}$ of size $|M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1$. For cardinality reason, there must be an element $x \in M$ whose image $f(x) = I$ is not in $\boldsymbol{S}$. But it is then impossible to have $x \in f(x)$, and f cannot be a witness of an embedding $M \in_{\text{emb}} \boldsymbol{B}$. Therefore, $M \notin {\downarrow_{\in}} \boldsymbol{B} \oplus D^{\circledast}$. □

The above procedure can obviously be implemented in exponential-time. Observe that this is already unavoidable in the case of $(X, \leq) = (\mathbb{N}^2, \leq_{\times})$: consider $\boldsymbol{B}_n = n \times \{\!|\, {\downarrow}\langle 1,1\rangle \,|\!\}$ and $D_n = {\downarrow}\langle 0,0\rangle$. Indeed, it is immediate to check that any multiset of size $n+1$ whose elements are among $\langle 0,1\rangle$ and $\langle 1,0\rangle$ is a minimal element of $X^{\circledast} \smallsetminus {\downarrow_{\in}} \boldsymbol{B}_n \oplus D_n^{\circledast}$. There are 2^n such multisets.

Below we present an optimization of the above procedure from which it is simple to derive a polynomial-time complexity implementation in the case of $(X, \leq) = (\mathbb{N}, \leq)$.

Proposition 7.1.10. *Given $\boldsymbol{B} \in (Idl(X, \leq))^{\circledast}$ and $D \in Down(X, \leq)$,*

$$X^{\circledast} \smallsetminus {\downarrow_{\in}} \boldsymbol{B} \oplus D^{\circledast} = \bigcup_{\substack{\boldsymbol{S} \subseteq Supp(\boldsymbol{B}) \\ G(\boldsymbol{S}) \text{ is connected}}} {\uparrow}\{M \mid Supp(M) \subseteq U_{\boldsymbol{S},\boldsymbol{B},D} \wedge |M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1\}$$

where $G(\boldsymbol{S}) = (\boldsymbol{S}, E)$ and $E = \{(I,J) \mid I \cap J \smallsetminus (D \cup \bigcup \boldsymbol{B} \smallsetminus \boldsymbol{S}) \neq \emptyset\}$

This proposition claims that it suffices to take the union over subsets $\boldsymbol{S}$ of $Supp(\boldsymbol{B})$ that induce a connected graph $G(\boldsymbol{S})$. Now in the case of $\mathbb{N}$, ideals of $\boldsymbol{B}$ can be sorted, and it suffices to consider subsets $\boldsymbol{S} \subseteq Supp(\boldsymbol{B})$ that are convex in the following sense: if $I, J \in \boldsymbol{S}$ and $K \in \boldsymbol{B}$ such that $I \subseteq K \subseteq J$, then $K \in \boldsymbol{S}$ as well. It is obvious that subsets $\boldsymbol{S}$ that are not convex induce a non connected graph $G(\boldsymbol{S})$. Thus it suffices to take the union over convex subsets. Moreover, there are only a quadratic (in $|\boldsymbol{B}|$) number of convex subsets of $Supp(\boldsymbol{B})$. Besides, since filters of $\mathbb{N}$ have at most one minimal elements, the expression builds at most one multiset per subset $\boldsymbol{S} \subseteq Supp(\boldsymbol{B})$. Therefore, $\mathbb{N}^{\circledast} \smallsetminus {\downarrow_{\in}} \boldsymbol{B} \oplus D^{\circledast}$ is computable in quadratic-time.

Proof. Let $\boldsymbol{S} \subseteq Supp(\boldsymbol{B})$ and M such that $Supp(M) \subseteq U_{\boldsymbol{S},\boldsymbol{B},D}$ and $|M| = |\boldsymbol{B}_{|\boldsymbol{S}}| + 1$. We show that if $G(\boldsymbol{S})$ is not connected then M is not minimal in $X^{\circledast} \smallsetminus {\downarrow_{\in}} \boldsymbol{B} \oplus D^{\circledast}$.

First of all, $Supp(M) \subseteq X \smallsetminus (D \cup \bigcup \boldsymbol{B} \smallsetminus \boldsymbol{S})$. Besides, if there exists some $x \in M$ such that $x \in X \smallsetminus (D \cup \bigcup \boldsymbol{B})$, then $\{\!|x|\!\} \leq_{\text{emb}} M$ and $\{\!|x|\!\} \notin {\downarrow_{\in}} \boldsymbol{B} \oplus D^{\circledast}$. This implies that M is not minimal, except if $|M| = 1$, which would imply $\boldsymbol{S} = \boldsymbol{\emptyset}$, which would imply $G(\boldsymbol{S})$ is connected, which is a contradiction. We can now assume that $Supp(M) \subseteq \bigcup \boldsymbol{S}$.

Take $\boldsymbol{S}_1 \sqcup \boldsymbol{S}_2$ a non-trivial partition of $\boldsymbol{S}$ such that $\boldsymbol{S}_1 \times \boldsymbol{S}_2 \cap E = \emptyset$, *i.e.* for all $(I,J) \in \boldsymbol{S}_1 \times \boldsymbol{S}_2$, $I \cap J \smallsetminus (D \cup \bigcup \boldsymbol{B} \smallsetminus \boldsymbol{S}) = \emptyset$. Hence, for every $x \in M$, either $x \in \bigcup \boldsymbol{S}_1$ or $x \in \bigcup \boldsymbol{S}_2$, the two options being mutually exclusive.

Define $M_i = M_{|\bigcup \boldsymbol{S}_i}$ for $i = 1, 2$. In particular, $M = M_1 + M_2$. Moreover, $M_i \in_{\text{emb}} \boldsymbol{B}$ implies $M_i \in_{\text{emb}} \boldsymbol{B}_{|\boldsymbol{S}_i}$. It is thus impossible that both M_1 and M_2 belong to $\downarrow_\in \boldsymbol{B} \oplus D^\circledast$: it would imply $M = M_1 + M_2 \in \downarrow_\in \boldsymbol{B} \oplus D^\circledast$. Without loss of generality, assume $M_1 \notin \downarrow_\in \boldsymbol{B} \oplus D^\circledast$. The elements of M_1 overall belong to only $m = |\boldsymbol{B}_{|\boldsymbol{S}_1}|$ ideals of $\boldsymbol{B}$. Thus, any submultiset N of M_1 with $m + 1$ elements is already not a member of $\downarrow_\in \boldsymbol{B} \oplus D^\circledast$, and since $|\boldsymbol{S}_1| < |\boldsymbol{S}|$, such multisets N are *strictly* smaller than M. □

Intersecting Filters (IF) **and Ideals** (II)

We now present procedures to intersect filters and ideals of $(X^\circledast, \leq_{\text{emb}})$. To emphasize the similarities between these two operations, we gather their description in a same proposition. Remember S_n stands for the group of permutations over $[n]$. The occurrence of S_n in the next formulas is no surprise, since S_n is at the heart of the equivalence relation $\sim$ from which multisets are built, and thus at the heart of functions $\mathcal{C}l_{\text{F}}$ and $\mathcal{C}l_{\text{I}}$. Yet, the two following formulas are less redundant than the expression one would directly get applying Section 4.2.

Proposition 7.1.11. *Given $M, N \in X^\circledast$, $\boldsymbol{B} \in Idl(X)^\circledast$ and $D \in Down(X)$:*

(IF)**:** *Intersection of filters:*

$$\uparrow M \cap \uparrow N = \uparrow \left\{ P + M_2 + N_2 \;\middle|\; \begin{array}{ll} M = M_1 + M_2, & |N_1| = |M_1| \\ N = N_1 + N_2, & P \in M_1 \tilde{\cap} N_1 \end{array} \right\}$$

where

$$\{\!|x_1 \cdots x_k|\!\} \tilde{\cap} \{\!|y_1 \cdots y_k|\!\} = \{\{\!|z_1 \cdots z_k|\!\} \mid \exists \sigma \in S_k.\, \forall i \in [k].\, z_i \in \min(\uparrow x_i \cap \uparrow y_{\sigma(i)})\}$$

(II)**:** *Intersection of ideals:*

$$\downarrow_\in \boldsymbol{B} \oplus D^\circledast \cap \downarrow_\in \boldsymbol{C} \oplus E^\circledast = \bigcup_{\substack{\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2 \\ \boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2 \\ |\boldsymbol{B}_1| = |\boldsymbol{B}_2|}} \downarrow_\in [\boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1 + \boldsymbol{B}_2 \sqcap E + \boldsymbol{C}_2 \sqcap D] \oplus (D \cap E)^\circledast$$

where $\{\!|I_1 \cdots I_k|\!\} \tilde{\cap} \{\!|J_1 \cdots J_k|\!\} = \bigcup_{\sigma \in S_k} \{\!|(I_1 \cap J_{\sigma(1)}) \cdots (I_k \cap J_{\sigma(k)})|\!\}$ *and* $\{\!|I_1 \cdots I_k|\!\} \sqcap D = \{\!|(I_1 \cap D) \cdots (I_k \cap D)|\!\}$

These expressions are computable in exponential-time (enumeration of $n!$ and $m!$ permutations). Here again, it is unavoidable in general, which is proved thereafter in Proposition 7.1.12.

Proof.

(IF): ($\supseteq$) It is immediate that every multiset $P + M_2 + N_2$ obtained as described in the proposition embeds both M and N.

($\subseteq$) Let $Q \in \uparrow M \cap \uparrow N$. There exist decompositions $Q = Q_1 + Q_2 + Q_3 + Q_4$, $M = M_1 + M_2$ and $N = N_1 + N_2$ such that:

- $|M_1| = |N_1| = |Q_1|$ and $M_1 \leq_{\text{emb}} Q_1$ and $N_1 \leq_{\text{emb}} Q_1$
- $|M_2| = |Q_2|$ and $M_2 \leq_{\text{emb}} Q_2$
- $|N_2| = |Q_3|$ and $N_2 \leq_{\text{emb}} Q_3$

Indeed, fix two embeddings f and g witnessing $M \leq_{\text{emb}} Q$ and $N \leq_{\text{emb}} Q$. Then take Q_1 to be the intersection of the images of f and g, while Q_2 is what remains of the image of f and Q_3 what remains of the image of g. Finally, Q_4 is what is outside both those images.

Now, by the first condition, Q_1 must be greater than some $P \in M_1 \tilde{\cap} N_1$, and thus $P + M_2 + N_2 \leq_{\text{emb}} Q_1 + Q_2 + Q_3 \leq_{\text{emb}} Q$.

(II): ($\supseteq$) Given decompositions $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$, $\boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2$ with $|\boldsymbol{B}_1| = |\boldsymbol{C}_1|$, we show $\downarrow_{\in}[\boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1 + \boldsymbol{B}_2 \sqcap E + \boldsymbol{C}_2 \sqcap D] \oplus (D \cap E)^{\circledast} \subseteq \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ using Proposition 7.1.5. The inclusion in $\downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}$ is analogous.

We have $\boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1 \subseteq_{\text{emb}} \boldsymbol{B}_1$ and $\boldsymbol{B}_2 \sqcap E \subseteq_{\text{emb}} \boldsymbol{B}_2$, thus

$$\begin{aligned}(\boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1 + \boldsymbol{B}_2 \sqcap E + \boldsymbol{C}_2 \sqcap D) \smallsetminus \mathit{Down}(D) &\subseteq_{\text{emb}} \boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1 + \boldsymbol{B}_2 \sqcap E \\ &\subseteq_{\text{emb}} \boldsymbol{B}_1 + \boldsymbol{B}_2 = \boldsymbol{B}\end{aligned}$$

Moreover, $D \cap E \subseteq D$.

($\subseteq$): Let $M \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}$. Define $M_1 = M \smallsetminus (D \cup E)$, $M_2 = M \smallsetminus (D \cup \complement E)$, $M_3 = M \smallsetminus (\complement D \cup E)$. Thus, $M \smallsetminus (D \cap E) = M_1 + M_2 + M_3$. Moreover, $M_1 + M_2 = M \smallsetminus D \in_{\text{emb}} \boldsymbol{B}$, hence we can decompose $\boldsymbol{B}$ in $\boldsymbol{B}_1 + \boldsymbol{B}_2$ such that $M_1 \in_{\text{emb}} \boldsymbol{B}_1$, $M_2 \in_{\text{emb}} \boldsymbol{B}_2$ and $|M_1| = |\boldsymbol{B}_1|$. Respectively, $M_1 + M_3 = M \smallsetminus E \in_{\text{emb}} \boldsymbol{C}$ and we define $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ similarly. Therefore, $|\boldsymbol{B}_1| = |M_1| = |\boldsymbol{C}_1|$ and $M_1 \in_{\text{emb}} \boldsymbol{B}_1 \tilde{\cap} \boldsymbol{C}_1$. Moreover, $\mathit{Supp}(M_2) \subseteq E$ and $M_2 \in_{\text{emb}} \boldsymbol{B}_2$, thus $M_2 \in_{\text{emb}} (\boldsymbol{B}_2 \sqcap E)$. Similarly $M_3 \in_{\text{emb}} (\boldsymbol{C}_2 \sqcap D)$, which concludes the proof. ☐

We now prove a lower bound matching the exponential upper bound given above. We show that intersections of filters and ideals are already exponential in $(\mathbb{N}^{2\circledast}, \leq_{\text{emb}})$. As stated above, it is obviously polynomial in $(A^{\circledast}, \leq_{\text{emb}})$, if A is a finite alphabet. It is also polynomial in $(\mathbb{N}^{\circledast}, \leq_{\text{emb}})$, which will be proved in Proposition 7.1.13.

Proposition 7.1.12. *Let* $(X, \leq) = (\mathbb{N}^2, \leq_{\times})$ *and* $n = 2m + 1$ *be an odd integer. Define* $M_n = \{\!|\langle n-1, 0\rangle \cdot \langle n-2, 1\rangle \cdots \langle m, m\rangle|\!\} = \sum_{i=0}^{m} \{\!|\langle n-i-1, i\rangle|\!\}$ *and* $N_n = \{\!|\langle 0, n-1\rangle \cdot \langle 1, n-2\rangle \cdots \langle m, m\rangle|\!\} = \sum_{i=0}^{m} \{\!|\langle i, n-i-1\rangle|\!\}$. *Then:*

(IF): *The upward-closed set* $\uparrow M_n \cap \uparrow N_n$ *has exponentially many (in* n*) minimal elements.*

(II)*: The downward-closed set* $\downarrow M_n \cap \downarrow N_n$ *has exponentially many (in* n*) maximal elements.*

It follows that both these operations require exponential-time computations.

Proof. Recall from Section 5.3 that $\uparrow\langle a, b\rangle \cap \uparrow\langle c, d\rangle = \uparrow\langle \max(a, c), \max(c, d)\rangle$ and $\downarrow\langle a, b\rangle \cap \downarrow\langle c, d\rangle = \downarrow\langle \min(a, c), \min(c, d)\rangle$. Given σ a permutation of $[0, m]$, define:

$$M_\sigma^\uparrow = \sum_{i=0}^{m} \{\!|\langle \max(n - i - 1, \sigma(i)), \max(i, n - \sigma(i) - 1)\rangle|\!\}$$

$$M_\sigma^\downarrow = \sum_{i=0}^{m} \{\!|\langle \min(n - i - 1, \sigma(i)), \min(i, n - \sigma(i) - 1)\rangle|\!\}$$

We now argue that multisets $M_\sigma^\uparrow$ are minimal elements of $\uparrow M_n \cap \uparrow N_n$ and $M_\sigma^\downarrow$ are maximal elements of $\downarrow M_n \cap \downarrow N_n$. This concludes the proof since these two families are exponential in n.

The multisets $M_\sigma^\uparrow$ are obviously members of $\uparrow M_n \cap \uparrow N_n$. Assume they are not minimal, then there exists a multiset P strictly smaller than some $M_\sigma^\uparrow$ which is in $\uparrow M_n \cap \uparrow N_n$. There are two possibilities for P to be *strictly* smaller: either it is shorter, *i.e.* $|P| < |M_\sigma^\uparrow| = m + 1$ but then $M_n \leq_{\mathrm{emb}} P$ is impossible since $|M_n| = m + 1$. Or $P \leq_{\mathrm{emb}} M_\sigma^\uparrow - \{\!|x|\!\} + \{\!|x - e_j|\!\}$ for some $x \in M_\sigma^\uparrow$ and some $j = 1, 2$ where $e_1 = \langle 1, 0\rangle$ and $e_2 = \langle 0, 1\rangle$. Without loss of generality, assume $j = 0$. Besides, $x = \langle \max(n - i - 1, \sigma(i)), \max(i, n - \sigma(i) - 1)\rangle$ for some $i \in [0, m]$. Depending on whether $i \geq n - \sigma(i) - 1$ this will either violates $M_n \leq_{\mathrm{emb}} P$ or $N_n \leq_{\mathrm{emb}} P$ considering the number of elements greater than $\max(i, n-\sigma(i)-1)$ in M_n or N_n. $\square$

Proposition 7.1.13. *There are polynomial-time procedures to intersect filters and ideals of* $(\mathbb{N}^\circledast, \leq_{\mathrm{emb}})$.

Proof.

(IF)**:** Observe that since $(\mathbb{N}, \leq)$ is linear, multisets of $\mathbb{N}^\circledast$ can be sorted. In particular, if $M = \{\!|x_1 \cdots x_n|\!\}$ and $N = \{\!|y_1 \cdots y_m|\!\}$ with $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_m$ and $n < m$ then $M \leq_{\mathrm{emb}} N$ if and only if $\forall i \in [n].\ x_i \leq y_i$.

Moreover, notice that if $|M| \leq |N|$, then $\uparrow M \cap \uparrow N = \uparrow(M + (|N| - |M|) \times \{\!|0|\!\}) \cap \uparrow N$, *i.e.* we can pad the shortest multiset with zeros. We can thus restrict our attention to multisets of the same length.

Given $M = \{\!|x_1 \cdots x_n|\!\}$ and $N = \{\!|y_1 \cdots y_n|\!\}$ with $x_1 > x_2 > \cdots > x_n$ and $y_1 > y_2 > \cdots > y_n$,

$$\uparrow M \cap \uparrow N = \uparrow\{\!|\max(x_1, y_1) \cdots \max(x_n, y_n)|\!\}$$

Indeed, if $\{\!|z_1 \cdots z_p|\!\} \in \uparrow M \cap \uparrow N$ with $z_1 > z_2 > \cdots > z_p$, then using the observation above, $z_i \geq \max(x_i, z_i)$ for $i \in [n]$.

(II): Here also, ideals of $\mathbb{N}$ are linearly ordered on the one hand, and ideals can be padded on the other hand. That is, we can restrict our attention to intersections $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}$ of ideals of the following shape:

- $\boldsymbol{B} = \{\!| I_1 \cdots I_n |\!\}$ with $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$,
- $\boldsymbol{C} = \{\!| J_1 \cdots J_m |\!\}$ with $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m$,
- We can assume $m = n$: assume otherwise suppose without loss of generality that $|\boldsymbol{B}| < |\boldsymbol{C}|$. We distinguish two cases:
  - Either $D \neq \emptyset$, in which case D is an ideal (all downward-closed sets of $\mathbb{N}$ are ideals, except $\emptyset$), in which case $\boldsymbol{B}$ can be padded with copies of D (*cf.* proof of Definition 7.1.6):

    $$\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} = \downarrow_{\in} [\boldsymbol{B} + (|\boldsymbol{C}| - |\boldsymbol{B}|) \times \{\!| D |\!\}] \oplus D^{\circledast}$$

    .
  - Or $D = \emptyset$, in which case all multisets $M \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ have length bounded by $n = |B|$. Thus

    $$\begin{aligned} \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast} &= \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap (\mathbb{N}^{\leq n} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}) \\ &= \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \{\!| J_1 \cdots J_n |\!\} \oplus E^{\circledast} \end{aligned}$$

    where $\mathbb{N}^{\leq n}$ designates the set of multisets of length at most n of $\mathbb{N}$. Indeed, for a multiset of length bounded by n to be in $\downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}$, only the n greatest ideals of $\boldsymbol{C}$ are relevant.
- For all $I \in \boldsymbol{B}$, $D \subseteq I$: this is a weak form of the canonical form described in Definition 7.1.6. Indeed, assuming canonical form for the ideal $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ would entail that for all $I \in \boldsymbol{B}$, $I \not\subseteq D$, *i.e.* $D \subsetneq I$. But because of the previous step of padding, it might be the case that $D \in \boldsymbol{B}$.
- Similarly, for all $J \in \boldsymbol{C}$, $E \subseteq J$.

It is clear that from any two ideals $\boldsymbol{I}, \boldsymbol{J}$ of $(\mathbb{N}^{\circledast}, \leq_{\mathrm{emb}})$ we can produce in polynomial-time two ideals $(\boldsymbol{I'}, \boldsymbol{J'})$ that have the specific form described above, and such that $\boldsymbol{I} \cap \boldsymbol{J} = \boldsymbol{I'} \cap \boldsymbol{J'}$. Now, for ideals $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ and $\downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast}$ satisfying the above conditions, we have:

$$\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast} = \downarrow_{\in} \{\!| (I_1 \cap J_1) \cdots (I_n \cap J_n) |\!\} \oplus (D \cap E)^{\circledast}$$

We prove this equation by induction on n. The base case is trivial since $D^{\circledast} \cap E^{\circledast} = (D \cap E)^{\circledast}$. For the inductive case, let $x \in \mathbb{N}$ and $M \in \mathbb{N}^{\circledast}$ such that $x \geq y$ for all $y \in M$, and let $I \in \mathit{Idl}(\mathbb{N})$ and $\boldsymbol{B} \in \mathit{Idl}(\mathbb{N})^{\circledast}$ such that $I \supseteq K$ for any $K \in \boldsymbol{B}$, and let $J \in \mathit{Idl}(\mathbb{N})$ and $\boldsymbol{C} \in \mathit{Idl}(\mathbb{N})^{\circledast}$ such that $J \supseteq K$ for any $K \in \boldsymbol{C}$. Finally, let $D, E \in \mathit{Down}(\mathbb{N})$ and assume that $|\boldsymbol{B}| = |\boldsymbol{C}|$, that $D \subseteq K$ for any $K \in \boldsymbol{B}$ and that $E \subseteq K$ for any $K \in \boldsymbol{C}$. We have:

$$\begin{aligned} &\{\!| x |\!\} + M \in \downarrow_{\in} [\{\!| I |\!\} + \boldsymbol{B}] \oplus D^{\circledast} \cap \downarrow_{\in} \{\!| J |\!\} + \boldsymbol{C} \oplus E^{\circledast} \\ &\qquad \Leftrightarrow x \in I \cap J \wedge M \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \cap \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast} \end{aligned}$$

Left to right implication follows from the fact I and J are greater than ideals from $\boldsymbol{B}$ and $\boldsymbol{C}$, and greater than D and E. Right to left implication holds since x is greater than any elements of M. Now by induction hypothesis, M is in $\downarrow_{\in}\{\!|(I_1 \cap J_1)\cdots(I_n \cap J_n)|\!\} \oplus (D \cap E)^{\circledast}$, assuming $\boldsymbol{B} = \{\!|I_1 \cdots I_n|\!\}$ where elements are sorted, and similarly for $\boldsymbol{C}$. Thus this is equivalent to $\{\!|x|\!\} + M \in \downarrow_{\in}\{\!|(I \cap J) \cdot (I_1 \cap J_1)\cdots(I_n \cap J_n)|\!\} \oplus (D \cap E)^{\circledast}$. □

7.1.4 Related Work

The WQO $(X^{\circledast}, \leq_{\mathrm{emb}})$ appears naturally to order configurations of some Petri Nets extensions. The structure of its ideals and its ideally effectiveness are used in [18] for a forward algorithm to semi-decide reachability in timed Petri Nets, and in [27] to derive tight complexity upper bounds of the coverability problem for ν-Petri Nets. In the first case, $(X, \leq)$ is a finite alphabet, while in the second case it is $(\mathbb{N}^k, \leq_{\times})$.

Here also, these results have been generalized to Noetherian spaces in [13].

7.2 Multisets under the Manna-Dershowitz Ordering

The *multiset ordering* $N \leq_{\mathrm{ms}} M$, on the set $X^{\circledast}$ of finite multisets over the WPO $(X, \leq)$, has been introduced by Manna and Dershowitz in 1979 in the context of proving termination for rewriting systems ([35]). It is also called *domination ordering*, or the *multiset extension of* $<$ (all terms would induce a risk of confusion with the quasi-orderings of Section 7.1). It is equivalently defined by one of the following statements, the second one being the original definition, while the first one comes from [36]:

(Def 1) $\forall x \in X.\ N(x) > M(x) \Rightarrow (\exists y > x.\ M(y) > N(y))$

(Def 2) $\exists S, T \in X^{\circledast}.\ S \subseteq M \ \wedge\ N = M - S + T \ \wedge\ S$ dominates T
where a multiset S dominates another multiset T if $\forall x \in T\ \exists y \in S.\ x < y$, see [35].

(Def 3) $M - N$ dominates $N - M$

(Def 4) $N = \{\!|x_1 \cdots x_n|\!\}, M = \{\!|y_1 \cdots y_m|\!\}, \exists f : [n] \to [m] :$
$\forall i \in [n].\ ((x_i < y_{f(i)}) \vee (x_i \leq y_{f(i)} \wedge \forall j \neq i.\ f(j) \neq f(i)))$.
Intuitively, $N \leq_{\mathrm{ms}} M$ if N embeds in M as in the previous section, but we don't require that the embedding is injective when an element is mapped to a strictly greater element.

Note that despite the notations, operations $+$ and $-$ on multisets do not act as inverse, and in particular are not associative. For instance $(A + B) - C \neq A + (B - C)$. In the definitions above, and in the remaining of this section, we adopt the convention of a left-association, *i.e.* we write $A + B - C$ for $(A + B) - C$. In particular, in the second definition, $M - S + T$ should be read $(M - S) + T$. With this convention, $A + B - C = A - (C - B) + (B - C)$.

It is known that $(X^{\circledast}, \leq_{\mathrm{ms}})$ is a linear ordering whenever $(X, \leq)$ is. More precisely, if $(X, \leq)$ is isomorphic to some ordinal α, then $(X^{\circledast}, \leq_{\mathrm{ms}})$ is isomorphic to ω^{α}.

Note that, as we did in Section 5.4, we require $(X, \leq)$ to be a well *partial*-order in order to avoid working with $\equiv$ instead of equality. Note that with this assumption, $(X^{\circledast}, \leq_{\mathrm{ms}})$ is a WPO as well.

Proof. Equivalence between the four definitions.

First observe that for any two multisets M and N, we have $N = M - (M - N) + (N - M)$ (with implicit left parenthesizing). Implication (3) $\Rightarrow$ (2) directly follow from this observation.

(1) $\Rightarrow$ (3) Let M and N as in the first definition. Let x in $N - M$. Since $0 < \max(0, N(x) - M(x))$, we have $N(x) > M(x)$. According to the first definition, this implies that there exists $y > x$ with $M(y) > N(y)$, which means $y \in M - N$. This shows $M - N$ dominates $N - M$.

(2) $\Rightarrow$ (1) Assume $N = M - S + T$ with $S \subseteq M$ and S dominates T. Let $x \in X$ such that $N(x) > M(x)$, that is $M(x) - S(x) + T(x) > M(x)$, *i.e.* $T(x) > S(x)$. In particular, $x \in T$, hence there exists $y \in S$ such that $x < y$. Take y maximal in S with this property (S is finite), then $T(y) = 0$. Indeed, if not, there must be $z \in S$ with $z > y$, which contradicts the maximality of y. Now since $T(y) = 0$, $N(y) = M(y) - S(y) < M(y)$ since $y \in S$, which proves $N \leq_{\mathrm{ms}} M$ for the first definition.

(2) $\Rightarrow$ (4) Let $N \leq_{\mathrm{ms}} M$ according to the second definition, there are $S, T \in X^{\circledast}$ such that $S \subseteq M$, $N = M - S + T$ and S dominates T. That is, we can write $N = \{\!|y_1 \cdots y_m \cdot t_1 \cdots t_n|\!\}$ and $M = \{\!|y_1 \cdots y_m \cdot s_1 \ldots s_r|\!\}$. Define $f : [m+n] \to [m+r]$ by $f(i) = \begin{cases} i & \text{if } i \leq m \\ \epsilon_i & \text{otherwise} \end{cases}$, where ϵ_i for $i \in [n]$ is such that $t_i < s_{\epsilon_i}$ (it exists by the domination hypothesis). It is obvious that f satisfies the requirements from the fourth definition, its restriction to $[m]$ is injective, and its restriction to $m + [n]$ is a domination.

(4) $\Rightarrow$ (2) Assume $N = \{\!|x_1 \cdots x_n|\!\} \leq_{\mathrm{ms}} M = \{\!|y_1 \cdots y_m|\!\}$, and let $f : [n] \to [m]$ be a function satisfying the requirements of the fourth definition. Define $S = \{\!|y_j \mid j \notin Im(f) \vee (\exists i \in [n].\ x_i < y_j = y_{f(i)})|\!\}$ and $T = \{\!|x_i \mid x_i < y_{f(i)}|\!\}$. Then obviously $S \subseteq M$. Moreover, $M - S = \{\!|y_j \mid \exists i \in [n].\ f(i) = j \wedge x_i = y_j|\!\}$, hence $N = M - S + T$ (indeed, every element in N is either equal to its image or strictly smaller). Finally, S dominates T, proving that $N \leq_{\mathrm{ms}} M$ according to the second definition.

□

From the first or third definition, it is obvious that $\leq_{\mathrm{ms}}$ can be decided in polynomial-time (more precisely in time $O(|N| \cdot |M|)$).

7.2.1 The Ideals of $(X^{\circledast}, \leq_{\mathrm{ms}})$

From the fourth definition, it is obvious that $\leq_{\mathrm{emb}} \subseteq \leq_{\mathrm{ms}}$. Hence, $\leq_{\mathrm{ms}}$ is a WQO (when X is) and by Proposition 4.1.1, $Idl(X^{\circledast}, \leq_{\mathrm{ms}}) = \{\downarrow_{\leq_{\mathrm{ms}}} \boldsymbol{I} \mid \boldsymbol{I} \in Idl(X^{\circledast}, \leq_{\mathrm{emb}})\}$. This helps proving the following proposition:

Proposition 7.2.1. (IR)*:* $Idl(X^{\circledast}, \leq_{\mathrm{ms}}) = \{\downarrow B + D^{\circledast} \mid B \in X^{\circledast}, D \in Down(X)\}$

Warning: unlike in the previous section, here B is a multiset of elements of X, not of ideals. Observe that $\downarrow B + D^{\circledast} = \{M_1 + M_2 \mid M_1 \leq_{\mathrm{ms}} B \wedge M_2 \in D^{\circledast}\} = \{M \mid M \setminus D \leq_{\mathrm{ms}} B\}$.

Proof. ($\supseteq$) We show that given $B \in X^{\circledast}$ and $D \in Down(X)$, $\downarrow B + D^{\circledast}$ is an ideal.

- It is downward-closed: let $N \leq_{\mathrm{ms}} M \in \downarrow B + D^{\circledast}$. Obviously, this implies that $N \setminus D \leq_{\mathrm{ms}} M \setminus D$ and by assumption $M \setminus D \leq_{\mathrm{ms}} B$. Thus, by transitivity, $N \setminus D \leq_{\mathrm{ms}} B$.
- It is directed: let $M_1, M_2 \in \downarrow B + D^{\circledast}$. Let $N = B + M_{1|D} + M_{2|D}$. Then $N \in \downarrow B + D^{\circledast}$ and $M_i \leq_{\mathrm{ms}} N$.

($\subseteq$) Let $\boldsymbol{J} \in Idl(X^{\circledast}, \leq_{\mathrm{ms}})$, $\boldsymbol{J} = \downarrow_{\leq_{\mathrm{ms}}} \boldsymbol{I}$ for some $\boldsymbol{I} \in Idl(X^{\circledast}, \leq_{\mathrm{emb}})$, and $\boldsymbol{I} = \downarrow_{\in} \boldsymbol{C} \oplus E^{\circledast} = \{M \mid M \setminus E \in_{\mathrm{emb}} \boldsymbol{C}\}$ for some $\boldsymbol{C} \in Idl(X)^{\circledast}$ and $E \in Down(X)$. Write $\boldsymbol{C}$ as $\{\!|I_1 \cdots I_k \cdot \downarrow x_1 \cdots \downarrow x_m|\!\}$ where the I_i's are limit ideals (*i.e.* not principal). Define $B = \{\!|x_1 \cdots x_m|\!\}$ and $D = E \cup I_1 \cup \cdots \cup I_k$. We show that $\boldsymbol{J} = \downarrow_{\leq_{\mathrm{ms}}} \boldsymbol{I} = \downarrow B + D^{\circledast}$.

- ($\subseteq$) It is simple to see that $\boldsymbol{I} \subseteq \downarrow B + D^{\circledast}$ and since $\downarrow B + D^{\circledast}$ is downward-closed, we have one inclusion.
- ($\supseteq$) To show the second inclusion, let $M \in \downarrow B + D^{\circledast}$. Since each ideal I_i is directed and unbounded, while M is finite, it is possible to pick in each I_i an element strictly greater than any of the elements in $M_{|I_i}$. Putting these elements together, we define a multiset $M' \in_{\mathrm{emb}} \{\!|I_1 \cdots I_k|\!\}$. Now the following holds: $M \leq_{\mathrm{ms}} B + M' + M_{|E}$ (*cf.* definition 4 for instance), and since $B + M' \in_{\mathrm{emb}} \boldsymbol{C}$, it proves that $M \in \downarrow_{\leq_{\mathrm{ms}}} \boldsymbol{I}$.

□

Notice that once again this representation is not unique, and we will provide a canonical representation for each ideal. This once again relies on an efficient characterization of inclusion.

Proposition 7.2.2. (ID)*:*

$$\downarrow B_1 + D_1^{\circledast} \subseteq \downarrow B_2 + D_2^{\circledast} \textit{ iff } B_1 \setminus D_2 \leq_{\mathrm{ms}} B_2 \textit{ and } D_1 \subseteq D_2 \cup \downarrow_{<} Supp(B_2 - B_1)$$

The multiset $B_1 \setminus D_2$ is computable in linear-time in the size of B_1, the set $Supp(B_2 - B_1)$ is finite, and thus its strict downward closure is also computable using $\downarrow_{<} S = \complement \uparrow S \cap \downarrow S$.

Proof. ($\Rightarrow$) First, $B_1 \in \downarrow B_1 + D_1^{\circledast} \subseteq \downarrow B_2 + D_2^{\circledast}$, thus $B_1 \setminus D_2 \leq_{\mathrm{ms}} B_2$.

For the second part, let $x \in D_1$, consider the multiset $M = B_1 \setminus D_2 + \{\!|x \cdots x|\!\}$ with $|B_2| + 1$ copies of x. Obviously, $M \in \downarrow B_1 + D_1^{\circledast} \subseteq \downarrow B_2 + D_2^{\circledast}$ and thus $M \setminus D_2 \leq_{\mathrm{ms}} B_2$. Now, if $x \notin D_2$, then $x \in M \setminus D_2$ and $M(x) = B_1(x) + |B_2| + 1 >$

$|B_2| \geq B_2(x)$, thus there exists $y > x$ such that $B_2(y) > M(y) = B_1(y)$. This implies that $y \in Supp(B_2 - B_1)$ and $x < y$.

($\Leftarrow$) Let $M \in \downarrow B_1 + D_1^{\circledast}$, *i.e.* $M \setminus D_1 \leq_{\mathrm{ms}} B_1$. We want to show $M \setminus D_2 \leq_{\mathrm{ms}} B_2$. Decompose M in two parts: $M \setminus D_2 = (M \setminus D_2) \setminus D_1 + (M \setminus D_2)_{|D_1}$. For the first part, $(M \setminus D_2) \setminus D_1 = (M \setminus D_1) \setminus D_2 \leq_{\mathrm{ms}} B_1 \setminus D_2$. Besides, by the third definition of $\leq_{\mathrm{ms}}$, $B_1 \setminus D_2 = B_2 - S + T$ for $S = B_2 - (B_1 \setminus D_2)$ and $T = (B_1 \setminus D_2) - B_2$, and S dominates T.

For the second part, observe that $D_2 \cup \downarrow_< Supp(B_2 - B_1) = D_2 \cup \downarrow_< Supp(B_2 - (B_1 \setminus D_2))$ and thus $D_1 \setminus D_2 \subseteq \downarrow_< Supp(S)$. It follows that $(M \setminus D_2)_{|D_1}$ is dominated by S, and we obtain:

$$\begin{aligned}
M \setminus D_2 &= (M \setminus D_2) \setminus D_1 + (M \setminus D_2)_{|D_1} \\
&\leq_{\mathrm{ms}} B_1 \setminus D_2 + (M \setminus D_2)_{|D_1} \\
&= B_2 - S + (T + (M \setminus D_2)_{|D_1}) \\
&\leq_{\mathrm{ms}} B_2
\end{aligned}$$

□

Proposition 7.2.3. *For every ideal $\boldsymbol{I}$ of $(X^{\circledast}, \leq_{\mathrm{ms}})$, there exists a unique representation $\boldsymbol{I} = \downarrow B + D^{\circledast}$ such that $B_{|D} = \emptyset$.*

Besides, this canonical representation *is computable in quadratic-time from any other representation.*

Proof. Given $\boldsymbol{I} = \downarrow B + D^{\circledast}$, the canonical representation of $\boldsymbol{I}$ is $\downarrow(B \setminus D) + D^{\circledast}$, which is obviously computable using $|B| \cdot |D|$ membership tests in $(X, \leq)$.

Now for uniqueness, assume $\downarrow B_1 + D_1^{\circledast} = \downarrow B_2 + D_2^{\circledast}$ and for $i \in \{1, 2\}$, $B_{i|D_i} = \emptyset$. By contradiction, assume there exists $x \in B_1 - B_2$, and assume x maximal.

- Either $x \in D_2$: but since $D_2 \subseteq D_1 \cup \downarrow_< Supp(B_1 - B_2)$ and $x \notin D_1$ by assumption, it implies that there exists $y > x$ such that $B_1(y) > B_2(y)$. This contradicts the maximality of x.

- Or $x \notin D_2$: then since $B_1 \setminus D_2 \leq_{\mathrm{ms}} B_2$, there must be $y > x$ such that $B_2(y) > B_1(y)$. Once again, we can pick y maximal.
  - Either $y \in D_1$: but $D_1 \subseteq D_2 \cup \downarrow_< Supp(B_2 - B_1)$, but by maximality, y cannot be in $\downarrow_< Supp(B_2 - B_1)$. On the other hand, y is in B_2, and thus cannot be in D_2 either. Contradiction.
  - Or $y \notin D_1$: and since $B_2 \setminus D_1 \leq_{\mathrm{ms}} B_1$, this implies the existence of $z > y > x$ such that $B_1(z) > B_2(z)$, contradicting the maximality of x.

Using the symmetry of the situation, we have proved $B_1 = B_2$. Thus $B_1 - B_2 = B_2 - B_1 = \emptyset$, which entails $D_1 = D_2$.

□

7.2.2 The Domination Ordering on Multisets is Ideally Effective

As mentioned above, the ordering $\leq_{\mathrm{ms}}$ extends $\leq_{\mathrm{emb}}$. However, functions $\mathcal{C}l_{\mathrm{F}}$ and $\mathcal{C}l_{\mathrm{I}}$ from Section 4.1 are not computable in general in this setting, and we cannot show ideal effectiveness relying on our results on extensions. As in the case of the lexicographic quasi-ordering (Section 5.4), this comes from the fact that we cannot in general decide whether an ideal is principal or not (in the proof of Proposition 7.2.1, we need to distinguish between principal and limit ideals). This is formally proved in Section 8.4.2. We will thus have to provide decision procedures for every operation. Computing principal ideals is trivial (PI), the ideal decomposition of $X^{\circledast}$ simply is $X^{\circledast}$ (XI), and the empty multiset is the unique minimal element of $X^{\circledast}$ (XF).

Complementing Filters (CF)

Proposition 7.2.4. *Given $N \in X^{\circledast}$ and $x \in X$, define $N_x = N_{|\uparrow x} - \{\!|x|\!\}$.*
The complement of $\uparrow N$ is given by:

$$X^{\circledast} \smallsetminus \uparrow N = \bigcup_{x \in Supp(N)} \downarrow N_x + (X \smallsetminus \uparrow x)^{\circledast}$$

From this proposition we easily derive a polynomial-time procedure to complement filters.

Proof. ($\subseteq$) Let $M \notin \uparrow N$, by negating the first definition, this is equivalent to

$$\exists x \in N.\ N(x) > M(x) \wedge \forall y > x.\ M(y) \leq N(y)$$

Then $M \in \downarrow N_x + (X \smallsetminus \uparrow x)^{\circledast}$ directly follows from the fact that $\forall y \in X.\ M_{|\uparrow x}(y) \leq N(y)$.

($\supseteq$) Let $x \in Supp(N)$ and $M \in \downarrow N_x + (X \smallsetminus \uparrow x)^{\circledast}$. We show that there exists y such that $N(y) > M(y)$ and for all $z > y$, $N(y) \geq M(y)$. Let S, T be two multisets such that $M_{|\uparrow x} = N_x - S + T$ and S dominates T. We consider two cases:

1. If x is maximal in S, then take $y = x$. By domination, for any $z \in T$, $z \not\geq x$, hence forall $z \geq x$, $M(z) = N_x(z)$. In particular, $M(x) = N_x(x) < N(x)$ and $M(z) \leq N_x(z) = N(z)$ for $z > x$.
2. Otherwise, take a maximal $y \in S$ such that $y \geq x$. In particular, $T(y) = 0$ and $S(y) \geq 1$, which implies that $M(y) = N_x(y) - S(y) + T(y) < N_x(y) \leq N(y)$. Besides, given $z > y$, $M(z) = N_x(z) - 0 + 0 \leq N(z)$.

□

Complementing Ideals (CI)

Proposition 7.2.5. *Let $B \in X^{\circledast}$ and $D \in Down(X)$.*
Given $x \in X$, define $B_x = B_{|\uparrow x} + \{\!|x|\!\}$
Then:

$$X^{\circledast} \smallsetminus (\downarrow B + D^{\circledast}) = \bigcup_{S \subseteq Supp(B)} \uparrow\{B_x \mid x \in \min(X \smallsetminus (D \cup \downarrow S))\}$$

The above expression is clearly computable, using (PI) and (CI) for $(X, \leq)$. However, it requires exponential-time to enumerate all the subsets of $Supp(B)$. This enumeration is justified unavoidable below. Note that this operation is computable in polynomial-time if $(X, \leq) = (\mathbb{N}^k, \leq_\times)$, which is proved thereafter.

Proof. ($\subseteq$) Let $M \notin \downarrow B + D^\circledast$. Let $N = M \setminus D$, by assumption $N \not\leq_{\rm ms} B$. Thus, there exists $x \in X$ such that $N(x) > B(x)$ and for all $y > x$, $N(y) \geq B(y)$ (negation of the first definition of $\leq_{\rm ms}$). Since $N(x) > 0$, $x \notin D$. Let S be the largest subset of $Supp(B)$ such that $x \notin \downarrow S$. It exists since $x \notin \emptyset$ and the property is stable by union. The element x is thus in $X \setminus (D \cup \downarrow S)$ and there exists $y \in \min(X \setminus (D \cup \downarrow S))$ such that $y \leq x$.

We now prove that $B_y \leq_{\rm ms} M$. Observe that, by maximality of S, for all $z \in Supp(B) \setminus S$, $x \leq z$. Thus, $B_{|\uparrow x} = B_{|\uparrow y}$ which proves that $B_y \leq_{\rm ms} B_x$. On the other hand, for any $z \in X$, $B_x(z) \leq M(z)$. This is immediate for $z \not\geq x$. For $z = x$, $M(x) = N(x) > B(x) = B_x(x) - 1$. And for $z > x$, $M(z) = N(z) \geq B(z) = B_x(z)$. This proves $B_y \leq_{\rm ms} B_x \leq_{\rm ms} M$ which concludes this direction of the proof.

($\supseteq$) For any $x \in X$, $B_x \not\leq_{\rm ms} B$: this is immediate by negating the first definition of $\leq_{\rm ms}$ and instantiating with x. Thus, for any $x \notin D$, $B_x \notin \downarrow B + D^\circledast$, which proves the desired inclusion.

□

We now provide a matching lower bound on the operation of complementing ideals, proving that the procedure given by Proposition 7.2.5 is asymptotically optimal in general. Notice that for a finite alphabet A, $(A^\circledast, \leq_{\rm emb}) = (A^\circledast, \leq_{\rm ms})$, since there are no elements $x, y \in A$ such that $x < y$. Thus, $(A^\circledast, \leq_{\rm ms}) \equiv (\mathbb{N}^{|A|}, \leq_\times)$ and all operations can be performed in polynomial-time. Besides, as stated in the introduction of this chapter, $(\mathbb{N}, \leq_{\rm ms})$ is isomorphic to the ordinal ω^ω, for which all operations are computable in polynomial-time as well.

The next natural candidate would be $\mathbb{N}^2$, but then it suffices to take union over subsets of size 2 of $Supp(B)$ in the formula of Proposition 7.2.5, which results in polynomial-time complexity. More generally:

Proposition 7.2.6. *For $(X, \leq) = (\mathbb{N}^k, \leq_\times)$, there is a polynomial-time procedure to complement ideals, described by the following expression: Given $B \in X^\circledast$ and $D \in Down(X)$,*

$$X^\circledast \setminus (\downarrow B + D^\circledast) = \bigcup_{\substack{S \subseteq Supp(B) \\ |S| \leq k}} \uparrow\{B_x \mid x \in \min(X \setminus (D \cup \downarrow S))\}$$

The only difference with the general expression is that it suffices to take the union over subsets of $Supp(B)$ that have size at most k.

Proof. We show that for every minimal elements x of $X \setminus (D \cup \downarrow S)$ for some $S \subseteq Supp(B)$ such that $|S| > k$, x is also a minimal element of $X \setminus (D \cup \downarrow S')$ for some $S' \subseteq Supp(B)$ with $|S'| \leq k$.

Indeed, denote by e_i the element of $\mathbb{N}^k$ that has a 1 on the i-th component, zeros elsewhere. Since x is minimal in $X \setminus (D \cup \downarrow S)$, for every $i \in [1, k]$, $x_i = x - e_i$

is either in D, or in $\downarrow S$, or $x_i \notin \mathbb{N}^k$ (if the i-th component of x is 0). Therefore, we can define a subset $S' \subset S$ of size at most k such that for every $i \in [1,k]$, $x_i \in D$ or $x_i \in \downarrow S'$, or $x_i \notin \mathbb{N}^k$. Obviously, x is still in $X \setminus (D \cup \downarrow S')$, and is still minimal since for every i, $x_i \in (D \cup \downarrow S')$ (if $x_i \in \mathbb{N}^k$).

□

Finally, the lower bound is proved using the polynomial-time ideally effective WQO $(\mathcal{P}_f(\mathbb{N}^2), \sqsubseteq_{\mathcal{H}})$ (*cf.* Section 7.3).

Proposition 7.2.7. *Let* $(X, \leq) = (\mathcal{P}_f(\mathbb{N}^2), \sqsubseteq_{\mathcal{H}})$ *where* $\mathbb{N}^2$ *is ordered with the product ordering. For* $n \in \mathbb{N}$, $i \in [1,n]$ *and* $U \subseteq [1,n]$*, define:*

$$\begin{aligned} x_i &= \langle i-1, n-i \rangle \\ S_U &= \{x_i \mid i \in U\} \\ T_i &= S_{[1,n] \setminus \{i\}} = \{x_j \mid j \neq i\} \\ B &= \{\!| T_1 \cdots T_n |\!\} \\ D &= \mathcal{P}_f(\bigcup_{i=1}^{n-1} \langle i-1, n-1-i \rangle) \end{aligned}$$

The upward-closed set $X^{\circledast} \setminus (\downarrow B + D^{\circledast})$ *has at least* $2^n - 1$ *minimal elements. In particular, any procedure for* (CI) *runs in exponential-time in the worst case.*

Proof. The elements $(x_i)_{1 \leq 1 \leq n}$ form an antichain of size n of $\mathbb{N}^2$. The downward-closed set D is chosen so that any set which is both smaller than $S_{[1,n]}$ and in $\complement D$ is equal to S_U for some $U \in [1,n]$.

Recall from Proposition 7.2.5 that minimal elements of $X^{\circledast} \setminus (\downarrow B + D^{\circledast})$ are of the form B_S, where $B_S = B_{|\uparrow S} + \{\!|S|\!\}$. We here show that multisets B_{S_U} all are minimal elements of $X^{\circledast} \setminus (\downarrow B + D^{\circledast})$ when U ranges over *strict* subsets of $[1,n]$. Given $U \subseteq [1,n]$, B_{S_U} will be denoted B_U for readability. Since $(x_i)_{1 \leq i \leq n}$ is an antichain, given $U, V \subseteq [1,n]$, $S_U \sqsubseteq_{\mathcal{H}} S_V \Leftrightarrow S_u \subseteq S_V \Leftrightarrow U \subseteq V$. Thus B_U has one copy of each T_i such that $i \notin U$, and an extra copy of S_U.

For any $U \subsetneq [1,n]$, $S_U \notin D$, thus $B_U \notin (\downarrow B + D^{\circledast})$ (*cf* second part of the proof of Proposition 7.2.5). It remains to show that each B_U is minimal. Let $U \subsetneq [1,n]$ and M be some multiset such that $M <_{\mathrm{ms}} B_U$. According to the second definition of $\leq_{\mathrm{ms}}$, there exists multisets P and Q such that $\emptyset \neq P \subseteq B_U$, $M = B_U - P + Q$ and P dominates Q. Without loss of generality, we can assume that $Supp(M) \cap D = \emptyset$. Subsequently, we show that $M \leq_{\mathrm{ms}} B$. Assume $M(S) > B(S)$ for some $S \in X$. By case analysis:

- If $S \in Q$, then there exists $T \in P$ such that $S \sqsubseteq_{\mathcal{H}} T$. In particular $T \in B_U$.

  1. If $T = S_U$: from $S \notin D$ and $S \sqsubseteq_{\mathcal{H}} S_U$, we deduce that $S = S_V$ for some $V \subsetneq U$ (D has been chosen for this property). Therefore, there exists $i \in U \setminus V$, hence $S = S_V \sqsubseteq_{\mathcal{H}} T_i$, while $T = S_u \not\sqsubseteq_{\mathcal{H}} T_i$. However, $B_U(R) = 0$ for any $T_i \sqsubseteq_{\mathcal{H}} R$, thus $M(T_i) = 0 < 1 \leq B(T_i)$.
  2. Otherwise, $T = T_i$ for some $i \notin U$, and we can assume that $S_U \neq T_i$. Then $M(T_i) = 0 < B(T_i) = 1$.

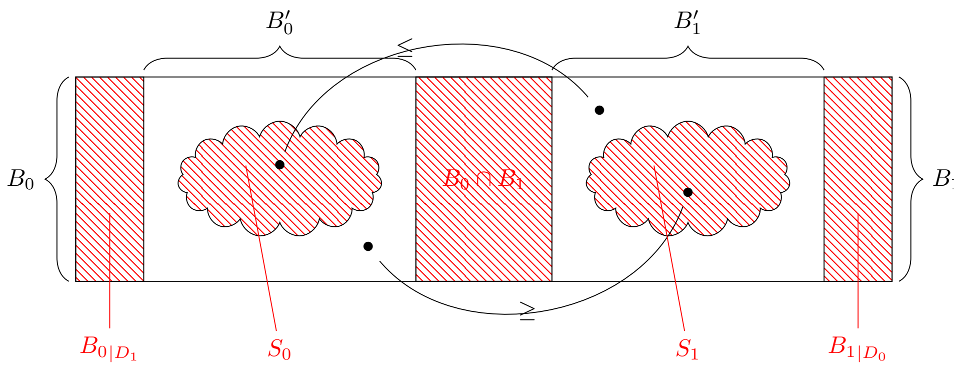

Figure 7.1: Intersection of Ideals

- If $x \notin Q$, then we have $B_U(S) \geq M(S) > B(S)$ and by definition of B_U this is only possible for $S = S_U$ in which case $B_U(S) = M(S) = B(S) + 1$. From $B_U(S) = M(S)$ we deduce that $S \notin P$, and P is not empty, thus there is some i such that $T_i \in P$. Note that this T_i cannot be equal to S_U since $B_U(S_U) = M(S_U)$. Thus, $S_U \sqsubset_H T_i$ and $M(T_i) = 0 < B(T_i) = 1$.

In conclusion, the exponential-time procedure described in Proposition 7.2.5 is optimal in the general.

□

Intersecting Ideals (II)

Ideals can be intersected using the following proposition (II).

Proposition 7.2.8. *Let* $\downarrow B_0 + D_0^{\circledast}$ *and* $\downarrow B_1 + D_1^{\circledast}$ *be two ideals given by their canonical representation, that is for* $i \in \{0, 1\}$, $B_{i|D_i} = \emptyset$. *For* $i \in \{0, 1\}$, *define* $\bar{\imath} = 1 - i$ *and:*

$$
\begin{aligned}
B'_i &= B_i \setminus D_{\bar{\imath}} - B_0 \cap B_1 \\
S_i &= \{x \in B'_i \mid \exists z \in B'_{\bar{\imath}}.\, z > x \wedge \forall t \in B'_i.\, t \not> z\} \\
B &= B_0 \cap B_1 + B'_{0|S_0} + B'_{1|S_1} + B_{0|D_1} + B_{1|D_0} \\
D &= \bigcup_{\substack{x \in B_0 - B \\ y \in B_1 - B}} (\downarrow x \cup D_0) \cap (\downarrow y \cup D_1)
\end{aligned}
$$

The intersection of two ideals is an ideal:

$$(\downarrow B_0 + D_0^{\circledast}) \cap (\downarrow B_1 + D_1^{\circledast}) = \downarrow B + D^{\circledast}$$

Fig. 7.1 may help the understanding of the above expression. The multiset B, depicted by the areas dashed in red on the figure, consists of the intersection $B_0 \cap B_1$, restrictions $B_{i|D_{\bar{\imath}}}$, plus the red clouds. Note that, as shown on the picture, these 3 parts of the definition of B are pairwise disjoint. Indeed, we assume that ideals $\downarrow B_i + D_i^{\circledast}$

are given by their canonical representations, hence there are no elements from D_i in B_i. Thus $B_0 \cap B_1 \cap B_{i|D_{\bar{\imath}}} \subseteq B_{\bar{\imath}|D_{\bar{\imath}}} = \emptyset$. The red clouds are disjoint from the two aforementioned parts by construction: we restrict our attention to what is left of the two multisets, which we denote B'_i. Of these multisets B'_i, we only keep in B elements from sets S_i. These sets consist of elements of B'_i that are dominated by elements in $B'_{\bar{\imath}}$ that are not themselves dominated in B'_i. In other words, for every $x \in S_i$ (point in the cloud), there exists some $y > x$ which is in $B'_{\bar{\imath}}$, but not in $S_{\bar{\imath}}$, *i.e.* not in the other cloud.

Once again, the procedure described above requires only polynomial-time.

Proof. First of all, notice from the construction that for any $x \in X$, $B(x) \in \{B_0(x), B_1(x)\}$. Indeed, consider the following cases:

- If $x \in D_0$: then $B_0(x) = 0$ since we assumed ideals given by their canonical representations. Thus $B(x) = \min(0, B_1(x)) + 0 + B'_{1|S_1}(x) + 0 + B_{1|D_0}(x)$. By definition, $B'_1(x) = 0$ and $B_{1|D_0}(x) = B_1(x)$. Thus $B(x) = B_1(x)$.
- Similarly if $x \in D_1$, $B(x) = B_0(x)$.
- Otherwise, $x \notin D_0 \cup D_1$. Note that $S_0 \cap S_1 = \emptyset$ by construction. Three cases remain:
  - If $x \in S_0$, then $x \notin S_1$ thus $B'_{0|S_0}(x) = B'_0(x)$ and $B'_{1|S_1}(x) = 0$. Thus, $B(x) = (B_0 \cap B_1)(x) + (B_0 \smallsetminus D_1)(x) - (B_0 \cap B_1)(x) = B_0(x)$.
  - Similarly if $x \in S_1$, $B(x) = B_1(x)$.
  - Otherwise, $B(x) = \min(B_0(x), B_1(x)) \in \{B_0(x), B_1(x)\}$.

($\subseteq$) Let $M \in (\downarrow B_0 + D_0^{\circledast}) \cap (\downarrow B_1 + D_1^{\circledast})$. We want to show $M \smallsetminus D \leq_{\mathrm{ms}} B$. Let $x \notin D$ such that $M(x) > B(x)$. We distinguish two cases:

1. Either $B(x) = \max(B_0(x), B_1(x))$. Four possibilities:
   (a) $x \notin D_0 \cup D_1$. In that case, since $M \smallsetminus D_0 \leq_{\mathrm{ms}} B_0$ and $M \smallsetminus D_1 \leq_{\mathrm{ms}} B_1$, there exist $y > x$ and $z > x$ such that $M(y) < B_0(y)$ and $M(z) < B_1(z)$. Now, if $B_0(y) > B(y)$ and $B_1(z) > B(z)$ hold simultaneously, then $x \in \downarrow y \cap \downarrow z \subseteq D$ which is a contradiction. Thus, at least one of the previous inequality does not hold, implying either $B(y) > M(y)$ or $B(z) > M(z)$.
   (b) $x \in D_0 \smallsetminus D_1$. In that case, we only have the existence of $z > x$ such that $B_1(z) > M(z)$. Again, if $B_1(z) > B(z)$ then $x \in \downarrow z \cap D_0 \subseteq D$, which is a contradiction. Thus $B(z) \geq B_1(z) > M(z)$.
   (c) $x \in D_1 \smallsetminus D_0$ is symmetrical.
   (d) Finally, $x \in D_0 \cap D_1$ is impossible, since $D_0 \cap D_1 \subseteq D$.
2. Or $B_i(x) > B(x) = B_{\bar{\imath}}(x)$. Without loss of generality, we assume $i = 0$. From $B_0(x) > B(x) \geq B_{0|D_1}(x)$ we deduce $x \notin D_1$. Since $M \smallsetminus D_1 \leq_{\mathrm{ms}} B_1$, there exists $y > x$ such that $B_1(y) > M(y)$. If $B_1(y) > B(y)$, then $y \in B_1 - B$, and since $x \in B_0 - B$, this would imply $x \in \downarrow x \cap \downarrow y \subseteq D$, which is a contradiction. Thus, $B(y) \geq B_1(y) > M(y)$.

($\supseteq$) We show $\downarrow B + D^{\circledast} \subseteq \downarrow B_i + D_i^{\circledast}$, for $i \in \{0,1\}$, using Proposition 7.2.2. We thus want to show that:

1. $B \setminus D_i \leq_{\mathrm{ms}} B_i$
2. $D \subseteq D_i \cup \downarrow_< Supp(B_i - B)$

We prove the result for $i = 0$, the case $i = 1$ being symmetrical. To prove the first point, assume some $x \notin D_0$ is such that $B(x) > B_0(x)$. By our preliminary remark, it implies $B(x) = B_1(x) > B_0(x)$, which also implies $x \in B_1$, and thus $x \notin D_1$. Now, since we are in the case $B(x) = \max(B_0(x), B_1(x)) = B_1(x)$, by definition $B'_{1|S_1}(x) \neq 0$, and thus: $x \in S_1$, *i.e.* $\exists z \in B'_0.\ z > x$, and $\nexists t \in B'_1.\ t > z$. From the last condition we deduce that $B'_{0|S_0}(z) = 0$, therefore $B(z) = \min(B_0(z), B_1(z))$. Furthermore, from $z \in B'_0$ we deduce that $B_0(z) > B_1(z)$, and conclude $B_0(z) > B(z)$.

For the second point, distributing the intersection over the unions leads to four cases to consider:

1. Given $x \in B_0 - B$ and $y \in B_1 - B$, $\downarrow x \cap \downarrow y \subseteq \downarrow_< Supp(B_0 - B)$ follows from $x \neq y$. This itself follows from $B_0 \cap B_1 \subseteq B$.
2. Given $x \in B_0 - B$, by construction of B, $x \notin D_1$. Thus for any z in $\downarrow x \cap D_1$, $z < x$, and $z \in \downarrow_< Supp(B_0 - B)$.
3. Given $y \in B_1 - B$, it is obvious that $\downarrow y \cap D_0 \subseteq D_0$.
4. Finally, $D_0 \cap D_1 \subseteq D_0$, obviously.

□

Intersecting Filters (IF)

Finally, the following proposition gives a polynomial-time procedure to intersect filters.

Proposition 7.2.9. *Given two multisets M_0 and M_1, and $i \in \{0,1\}$, we write $\bar{\imath}$ for $1 - i$, and define $S_i = \{x \in M_i \mid \forall z \in M_{\bar{\imath}} - M_i.\ z > x \Rightarrow \exists t \in M_i - M_{\bar{\imath}}.\ t > z\}$.*

The intersections of two filters is a filter:

$$\uparrow M_0 \cap \uparrow M_1 = \uparrow(M_{0|S_0} \cup M_{1|S_1})$$

It is easier to understand this complicated condition on examples. Take $X = \mathbb{N}^2$. If $M_0 = [(1,1)(2,1)]$ and $M_1 = [(1,2)]$, then $M_{0|S_0} \cup M_{1|S_1} = [(1,2)(2,1)]$, which is indeed greater than both M_0 and M_1. On the other hand, if $M_0 = [(1,1)(2,2)]$ and $M_1 = [(2,2)]$, then $M_{0|S_0} \cup M_{1|S_1} = [(1,1)(2,2)]$. Notice how $(1,1)$ is not part of the final result in the second example, while it is in the second. Intuitively, the condition expresses that an element x from M_0 will not be in the final result if it is dominated by some $z \in M_1$ such that z is in the final result. The fact that z is in the final result is expressed by negating the existence of a t in M_0 that dominates z. The condition on the multiplicities of z and t becomes clear when considering the case $M_0 = [(1,1)(2,2)]$ and $M_1 = [(2,2)(2,2)]$ and comparing it to the second example.

Proof. Define $M'_i = M_{i|S_i}$ and $M = M'_0 \cup M'_1$, that is for all $x \in X$, $M(x) = \max(M'_0(x), M'_1(x))$.

($\subseteq$) Let $N \in \uparrow M_0 \cap \uparrow M_1$. To prove $M \leq_{\text{ms}} N$, assume $N(x) < M(x) = \max(M'_1(x), M'_2(x))$ for some x. This implies that $M'_i(x) > N(x)$ for some $i \in \{0, 1\}$, and without loss of generality, we assume $i = 0$. This implies $M'_0(x) > 0$, thus $x \in S_0$ and $M'_0(x) = M_0(x)$. Since $M_0 \leq_{\text{ms}} N$, there exists $y > x$ such that $M_0(y) < N(y)$.

If $M'_1(y) < N(y)$, then $M(y) < N(y)$, which concludes the proof. We now prove that the other case, $M'_1(y) \geq N(y)$ is impossible. It in particular implies $y \in S_1$ and thus $M_1(y) \geq N(y) > M_0(y)$. Recall that $x \in S_0$, therefore $\forall z \in M_1 - M_0.\, z > x \Rightarrow \exists t \in M_0 - M_1.\, t > z)$. Instantiated with $z = y$, it gives the existence of $t \in M_0 - M_1$ such that $t > y$. Besides, we can consider this t maximal. Similarly, we instantiate with $z = t$ in the condition of $y \in S_1$, to obtain the existence of $u \in M_1 - M_0$ such that $u > t$. Now, we instantiate one more time the condition given by $x \in S_0$, but with $z = u$. This gives the existence of $v \in M_0 - M_1$ such that $v > u > t$, which contradicts the maximality of t. Thus this case is impossible, and $M \leq_{\text{ms}} N$.

($\supseteq$) We show that $M_0 \leq_{\text{ms}} M$, the other case being symmetrical. Assume $M_0(x) > M(x) = \max(M'_0(x), M'_1(x))$. This implies that $M'_0(x) = 0$ and $M'_1(x) < M_0(x)$. From $M'_0(x) = 0$, we deduce the existence of $z \in M_1 - M_0$ such that $z > x$ and for any $t \in M_0 - M_1, t \leq z$. This last part implies that $z \in S_1$, thus $M'_1(z) = M_1(z)$ and therefore $M(z) = \max(M'_0(z), M'_1(z)) = M'_1(z) = M_1(z) > M_1(z)$. $\square$

7.3 Finitary powerset over X

When $(X, \leq)$ is a QO, a natural quasi-ordering on $\mathcal{P}(X)$, the powerset over X, is the *Hoare quasi-ordering* (also called *majoring quasi-ordering*), denoted $\sqsubseteq_H$, and defined by

$$S \sqsubseteq_H T \overset{\text{def}}{\Leftrightarrow} \forall x \in S : \exists y \in T : x \leq y.$$

A convenient characterization of this quasi-ordering is the following: $S \sqsubseteq_H T$ iff $S \subseteq \downarrow_X T$. Note that $(\mathcal{P}(X), \sqsubseteq_H)$ is in general not antisymmetric even when $(X, \leq)$ is. For example $S \equiv_H \downarrow_X S$ for any $S \subseteq X$. Actually, $(\mathcal{P}(X)/\equiv_H, \sqsubseteq_{\mathcal{H}})$ is order-isomorphic to $(Down(X), \subseteq)$. It is sometimes stated that $(X, \leq)$ is a WQO if and only if $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$ is well-founded. With the above remark, this is exactly (WQO7) from the definition of WQO we gave in Chapter 2. In particular, $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$ needs not be a WQO, see Section 9.2 for more details. However, $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{H}})$ is a WQO, where $\mathcal{P}_f(X)$ denotes the set of all *finite subsets* of X. Indeed, it is the quotient of $(X^*, \leq_{\text{st}})$ (see Section 6.2) by the relation $\simeq$ defined by $\boldsymbol{u} \simeq \boldsymbol{v}$ iff $Supp(\boldsymbol{u}) = Supp(\boldsymbol{v})$, where $Supp(\boldsymbol{u})$ is the finite set of elements of X that appear in the finite sequence $\boldsymbol{u} \in X^*$. Moreover, $\simeq \circ \leq_{\text{st}} = \leq_{\text{st}} \circ \simeq$, and Section 4.2 applies. However, procedures for this WQO are rather simple to obtain directly, and more efficient that if relying on Section 4.2. Finite sets of X are once again represented as the explicit list of their elements (XR). The ordering $S \sqsubseteq_{\mathcal{H}} T$ can be tested using at most $|S| \cdot |T|$ comparisons of elements of S and T (OD).

7.3.1 The ideals of $\mathcal{P}_f(X)$

As mentioned above, $(\mathcal{P}_f(X), \sqsubseteq_H)$ can be obtained as a quotient of $(X^*, \leq_{\mathrm{st}})$ and thus the ideals of $(\mathcal{P}_f(X), \sqsubseteq_H)$ are exactly the closure under $\simeq$ of the ideals of $(X^*, \leq_{\mathrm{st}})$. The latter has been shown to be sets of the form $D_1^* \cdots D_k^*$ (*cf.* Section 6.2), and thus their closure under $\simeq$ is $(D_1 \cup \cdots \cup D_k)^*$. In terms of sets, this is exactly $\mathcal{P}_f(D_1 \cup \cdots \cup D_k)$, the set of finite subsets of $D_1 \cup \cdots \cup D_k$.

Lemma 7.3.1. (IR)*: The ideals of $(\mathcal{P}_f(X), \sqsubseteq_H)$ are exactly the sets of the form $\mathcal{P}_f(D)$, where D is a downward-closed subset of X.*

Proof. In complement of the sketch of proof that precedes the lemma, we present below a direct and simple proof:

($\Leftarrow$) : $\emptyset \in \mathcal{P}_f(D)$, so $\mathcal{P}_f(D)$ is nonempty. It is downward-closed, since if $S \sqsubseteq_H T \in \mathcal{P}_f(D)$, then $S \subseteq \downarrow_X T \subseteq \downarrow_X D = D$. It is directed, since if $S, T \in \mathcal{P}_f(D)$, then $S \cup T \in \mathcal{P}_f(D)$.

($\Rightarrow$) : Let $\mathcal{J}$ be an ideal of $\mathcal{P}_f(X)$. Let $D = \bigcup \mathcal{J} = \bigcup_{S \in \mathcal{J}} S$. Then clearly $\mathcal{J} \subseteq \mathcal{P}_f(D)$. Since $\mathcal{J}$ is downward-closed under $\sqsubseteq_H$, D is downward-closed under $\leq$ and $\{x\} \in \mathcal{J}$ for all $x \in D$. Since $\mathcal{J}$ is nonempty (it is an ideal), $\emptyset \in \mathcal{J}$. Finally, if $S, T \in \mathcal{J}$, then there is some $U \in \mathcal{J}$ such that $S, T \sqsubseteq_H U$. Thus $S \cup T \sqsubseteq_H U$, and $S \cup T \in \mathcal{J}$. Therefore, $\mathcal{J}$ has the empty set and singletons, and is closed under finite unions, and so is equal to $\mathcal{P}_f(D)$. □

Therefore, we represent ideals of $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{H}})$ as downward-closed sets of $(X, \leq)$, and "remember" that they stand for the set of all finite subsets of these downward-closed sets. Hence, checking inclusion $\mathcal{P}_f(D_1) \subseteq \mathcal{P}_f(D_2)$ boils down to inclusion $D_1 \subseteq D_2$ of downward-closed sets of $(X, \leq)$ (ID). The question of the uniqueness of the representation of a given ideal is also easily settled: we know downward-closed sets admit a canonical ideal decomposition, which is unique up to permutation. If ideals of $(X, \leq)$ have a unique representation themselves, then so have the ideals of $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{H}})$.

7.3.2 The Hoare Quasi-Ordering is Ideally Effective

Given $S \in \mathcal{P}_f(X)$, the notation $\downarrow S$ could represent the downward-closure of S as a subset of X, or the downward-closure of $\{S\}$ as a subset of $\mathcal{P}_f(X)$. We therefore annotate every occurence of a closure: $\downarrow_X S$ denotes the downward-closure of S as a subset of X, while $\downarrow_H S$ denotes the downward-closure of S in $\mathcal{P}_f(X)$.

Procedures for set-theoretic operations in $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{H}})$ are rather simple:

(XI)**:** The whole set $\mathcal{P}_f(X)$ is an ideal since X is downward-closed.

(XF)**:** The filter decomposition of $\mathcal{P}_f(X)$ is the empty set: $\uparrow_H \emptyset$ (not to be confused with the empty filter decomposition, which denotes the empty upward-closed set of $\mathcal{P}_f(X)$).

(PI)**:** Given $S \in \mathcal{P}_f(X)$, $\downarrow_H S = \mathcal{P}_f(\downarrow_X S)$ (obvious when considering the alternative definition of $\sqsubseteq_H$ given at the beginning of this chapter). Note that $\downarrow_X S = \bigcup_{x \in S} \downarrow x$ is computable.

(CF): Given $S \in \mathcal{P}_f(X)$, the complement of $\uparrow_H S$ is given by:

$$\begin{aligned}\mathcal{P}_f(X) \smallsetminus \uparrow_H S &= \bigcup_{x \in S} \mathcal{P}_f(X) \smallsetminus \uparrow_H \{x\} \\ &= \bigcup_{x \in S} \mathcal{P}_f(X \smallsetminus \uparrow x)\end{aligned}$$

(II): To intersect ideals: $\mathcal{P}_f(D_1) \cap \mathcal{P}_f(D_2) = \mathcal{P}_f(D_1 \cap D_2)$.

(IF): Filters may be intersected using $\uparrow S \cap \uparrow T = \uparrow(S \cup T)$.

(CI): Given D a downward-closed set of X, $\mathcal{P}_f(X) \smallsetminus \mathcal{P}_f(D)$ consists of the set that contain at least one element not in D. That is:

$$\mathcal{P}_f(X) \smallsetminus \mathcal{P}_f(D) = \uparrow_H \{\{x\} \mid x \in \min(X \smallsetminus D)\}$$

All procedures presented in this section can be performed in polynomial-time, in the sense given at the end of Chapter 3.

Chapter 8

A Minimal Set of Axioms

8.1 A Shorter Definition

Definition 3.1.1 obviously contains some redundancies. For instance, ideal membership (IM) is simply obtained using (PI) and (ID): $x \in I$ iff $\downarrow x \subseteq I$. We can thus shorten our main definition. An effective WQO $(X, \leq)$ further equipped with procedures for (CF), (II), (PI) and (XI) will be called a *short presentation* of $(X, \leq)$

In the following proposition we show that we only need to assume these four axioms to obtain an equivalent definition. Moreover these four axioms form a minimal system of axioms (*cf.* Proposition 8.1.3).

Proposition 8.1.1. *A full presentation of* $(X, \leq)$ *can be computed from a short presentation of* $(X, \leq)$.

Proof. We explain how to obtain the missing procedures:

(IM)**:** As mentioned above, membership can be tested using (PI) and (ID): $x \in I$ iff $\downarrow x \subseteq I$.

(CI)**:** We actually show a stronger statement, denoted CD, that complementing an arbitrary downward-closed set is computable. This strengthening is necessary for (IF).

Let D be an arbitrary downward-closed set. We compute $\complement D$ as follows:

1. Initialize $U := \emptyset$;
2. While $\complement U \not\subseteq D$ do
   (a) pick some $x \in \complement U \cap \complement D$;
   (b) set $U := U \cup \uparrow x$

Every step of this high-level algorithm is computable. The complement $\complement U$ is computed using the procedure to complement filters composed with the procedure to intersect ideals: $\complement \bigcup_{i=1}^{n} \uparrow x_i = \bigcap_{i=1}^{n} \complement \uparrow x_i$ which is computed with (CF) and (II) (or with (XI) in case $n = 0$, i.e., for $U = \emptyset$). Then, inclusion $\complement U \subseteq D$ is tested with (ID). If this test fails, then we know $\complement U \cap \complement D$ is not empty, and thus we can enumerate elements $x \in X$ by brute force, and test membership in U and in D. Eventually, we will find some $x \in \complement U \cap \complement D$.

To prove partial correctness we use the following loop invariant: U is upward-closed and $U \subseteq \complement D$. The invariant holds at initialization and is preserved by the loop's body since if $\uparrow x$ is upward-closed and since $x \notin D$ and D downward-closed imply $\uparrow X \subseteq \complement D$. Thus when/if the loop terminates, one has both $\complement U \subseteq D$ and the invariant $U \subseteq \complement D$, i.e., $U = \complement D$.

Finally, the algorithm terminates since it builds a strictly increasing sequence of upward-closed sets, which must be finite by Eq. (WQO6).

(IF)**:** This follows from (CF) and CD, by expressing intersection in terms of complement and union.

(XF)**:** Using CD we can compute $\complement\emptyset$.

□

Remark 8.1.2 (On Proposition 8.1.1)**.** *The above methods are generic but in many cases there are simpler and more efficient ways of implementing* (CI)*,* (IF)*, etc. for a given WQO. This is why Definition 3.1.4 lists eight requirements instead of just four: we wanted to provide efficient procedures for all main operations in concrete cases.*

As seen in the above proof, the fact that (CF), (II), (PI) and (XI) entail (CI) is non-trivial. The algorithm for CD computes an upward-closed set U from an oracle answering queries of the form "Is $U \cap I$ empty?" for ideals I. This is an instance of the Generalized Valk-Jantzen Lemma [37], an important tool for showing that some upward-closed sets are computable.

The existence of such a non-trivial redundancy in our definition led us to the question of whether there are other redundancies. The following proposition answers negatively.

Proposition 8.1.3. *There are no generic and computable ways to produce a procedure for axiom A given procedures for axioms B, C and D, where $\{A, B, C, D\} = \{\text{(CF)}, \text{(II)}, \text{(PI)}, \text{(XI)}\}$.*

This proposition is proved in the next section.

8.2 Proof of Proposition 8.1.3

In this section and the next ones, we describe several examples of WQOs that are not ideally effective, that is, some of the operations described in Chapter 3 are not computable. Observe that in Definition 3.1.1, the representations of elements and ideals is existentially quantified, hence we should theoretically prove that *for any* representations of the elements and ideals, some operation is not computable. Instead, we only prove the weaker statement that for some given reasonable representation, some operation is not computable.

To prove non-computability of an operation, we always reduce the halting problem for Turing machines. More precisely, from now on we fix an enumeration $(T_i)_{i\in\mathbb{N}}$ of Turing machines, such that there is a universal Turing Machine that can simulate T_i when given i in input. We use the following version of the halting problem: given i, decide whether T_i halts on the empty input (*i.e.* starting with an empty tape). This problem is well-known to be undecidable. Also define $t_i \in \mathbb{N} \cup \{\omega\}$ to be the halting time of machine T_i (on the empty input). For illustration purposes, we will assume that our enumeration is such that T_0 and T_1 halt ($t_0 < \omega$, $t_1 < \omega$) but not T_2 ($t_2 = \omega$).

All the (non ideally effective) WQOs given subsequently are built along the same idea: an element of the WQO intuitively corresponds to some execution step of some Turing machine T_i. Elements are ordered such that an element corresponding to step t of the execution of machine T_i is greater or equal to an element corresponding to step t' of the execution of machine T_j whenever $i > j$ or $i = j$ and $t' > t$. The simplest WQO satisfying these conditions is the ordinal ω^2 (equivalently, the lexicographic ordering over $\mathbb{N}^2$). Non ideally effective WQOs presented in this section are often obtained by modifying the shape of ω^2 around elements corresponding to steps t_i (halting steps), so that some set-theoretic operation on the WQO is able to spot the difference: it is then possible to compute t_i using an oracle for this operation, which proves the operation to be non computable, and thus the WQO to be non ideally effective. This is often done by quotienting the natural ordering on the ordinal ω^2 by a well-chosen equivalence relation. Therefore, it is advised to read Section 3.2.3 and Section 4.2 before the remainder of this chapter.

In all our examples, it is crucial that despite being undecidable, the halting problem is semi-decidable. In particular, given a time t and a machine number i, one can decide whether $t \leq t_i$. It suffices to simulate the machine T_i for t steps: if it halted before, then $t_i < t$, otherwise $t \leq t_i$.

We now proceed to the proof of Proposition 8.1.3. If $A \in \{\text{(CF)}, \text{(II)}\,\text{(PI)}\}$, we prove the following stronger statement: not only a procedure for A cannot be uniformly computed from procedures for the three remaining operations, but we provide a particular WQO for which A is not computable, while all other operations are. Such a stronger statement makes no sense for axiom (XI) since for a fixed WQO X, the ideal decomposition of the full set X is a constant, hence computable.

8.2.1 Complementing Filters (CF)

Define $X = \omega^2$ equipped with the quasi-ordering $\leq_E = \leq \circ E$ where $\leq$ is the natural ordering on ω^2 and E is the smallest equivalence relation such that:

$$(\omega \cdot i + t)E(\omega \cdot i + t + 1) \text{ when } t \neq t_i$$

(see Fig. 8.1: an edge between two points means these two elements are equivalent with respect to $\leq_E$, otherwise, greater elements are drawn above smaller elements).

Intuitively, associate to each Turing Machine T_i a copy of $\mathbb{N}$ where each natural number represents an execution time of the machine T_i. The WQO X is made of ω copies (one per Turing machine) on top of one another: this is a countable lexicographic sum, otherwise seen as the lexicographic product of ω by $\mathbb{N}$. Then, the equivalence relation E is gluing some elements together in each copy of $\mathbb{N}$ so that each copy has only one or two equivalence class(es): the class of all elements strictly below t_i, and if t_i is finite, the class of all elements above. Therefore, the copy associated with T_i has two equivalence classes if and only if T_i halts (and one otherwise).

We now show that the WQO described above is almost ideally effective: only (CF) cannot be computed. This proves that in general, (CF) cannot be computed from other operations.

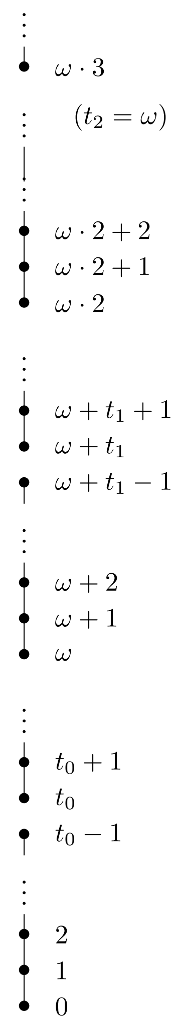

Figure 8.1: WQO for (CF) and (PI)

(XR): Elements of X are represented by tuples $\langle i, t \rangle \in \mathbb{N}^2$, standing for the ordinal $\omega \cdot i + t \in \omega^2$.

(OD): The quasi-ordering is decidable: $\omega \cdot i + t \leq_E \omega \cdot i' + t'$ iff $i < i'$; or $i = i'$ and $t, t' < t_i$; or $i = i'$ and $t_i \leq t, t'$. Conditions $t, t' < t_i$ and $t_i \leq t, t'$ are decidable by simulating T_i for $\max(t, t')$ steps.

(IR): Since each copy has only one or two equivalence class(es), the only limit ideal is X itself. We thus represent ideals by the elements of X, plus the special case of X.Note that this is not the representation we used for ordinals in Section 3.2.3.

(ID): Inclusion is decidable since it is essentially the same as the quasi-ordering, with the extra element ω^2 which is greater than any other.

(PI): The procedure for principal ideals is then trivial.

(II): Ideals are linearly quasi-ordered by inclusion, and thus, intersection consists of taking the minimum for inclusion.

(XI): The ideal decomposition of X is trivial since we made it a special case.

Now assume (CF), that is complement of filters are computable: on input $X \smallsetminus \uparrow(\omega \cdot (i+1))$ it must return some $\downarrow(\omega \cdot i + t)$. For the procedure for (CF) to be correct, it must be the case that $\downarrow(\omega \cdot i + t) = \downarrow(\omega \cdot i + t')$ for all $t' \geq t$, *i.e.* $\omega \cdot i + t$ and $\omega \cdot i + t'$ are equivalent. Thus, $t_i < \omega$ if and only if $t_i \leq t$, and the halting problem could be decided.

8.2.2 Principal Ideals (PI)

For (PI), consider the same WQO $(X, \leq_E)$ as before. However, we change the representation for ideals. Indeed, if the ordinals α and β are E-equivalent, then $\downarrow \alpha$ and $\downarrow \beta$ are two representations of the same ordinal. We thus make representations of ideals unique by allowing only two types of ideals:

- the limit ordinals of $\boldsymbol{\omega^2 + 1}$, that is $\boldsymbol{\omega^2} = X$ and $\boldsymbol{\omega \cdot i} = \{\alpha \mid \alpha < \omega \cdot i\}$ for $i \in \mathbb{N}$;
- and the successor ordinals of the form $\boldsymbol{\omega \cdot i + t_i}$ for $i \in \mathbb{N}$.

The set of ideals is still recursive, since given i and t, it is possible to check whether $t = t_i$. Ideal inclusion corresponds to a suborder of the natural ordering on ordinals, and is thus decidable. The inclusion on ideals being linear, intersecting ideals again corresponds to taking the minimum, and hence is computable. The ideal decomposition of X is still the same.

However, we now have a procedure for (CF):

- $X \smallsetminus \uparrow(\omega \cdot i + t) = \boldsymbol{\omega \cdot i}$ if $t < t_i$, which can be tested by simulating T_i for t steps.
- $X \smallsetminus \uparrow(\omega \cdot i + t) = \boldsymbol{\omega \cdot i + t_i}$ if $t \geq t_i$, which can similarly be tested. Note that if concluding $t \geq t_i$, then t_i has been found, and thus it can be output.

But there are no procedures for (PI) anymore: if there were, one could compute $\downarrow \omega \cdot i$ that should be mapped to $\boldsymbol{\omega \cdot i + t_i}$ if T_i halts, and to $\boldsymbol{\omega \cdot (i+1)}$ otherwise.

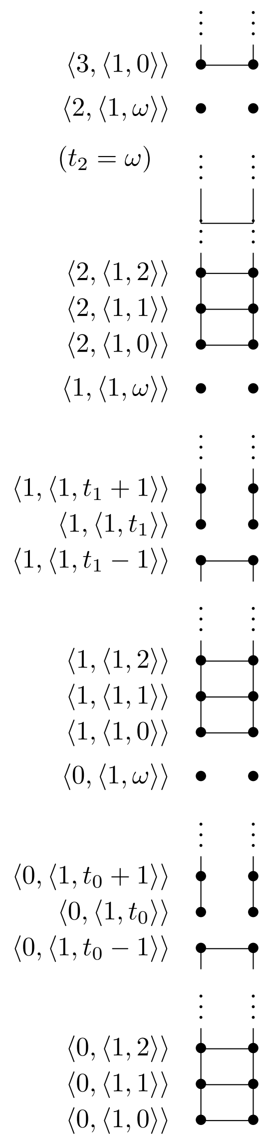

Figure 8.2: WQO for (II)

8.2.3 Intersecting Ideals (II)

Let $Y = (\mathbb{N} \sqcup \mathbb{N}) \oplus (\mathbf{1} \sqcup \mathbf{1})$ quasi-ordered with the sum quasi-orderings introduced in Chapter 5. The set Y consists of two copies of $\mathbb{N}$, augmented with two top elements that will be denoted $\langle 1, \omega \rangle$ and $\langle 2, \omega \rangle$, that are incomparable with each other, but greater than any other element of Y. In particular, note that for any $a \in \{1, 2\}$ and $n \in \mathbb{N}$, $\langle i, \langle a, \omega \rangle \rangle > \langle i, \langle 3-a, n \rangle \rangle$. In other words, Y is not quasi-ordered as $\boldsymbol{\omega+1} \sqcup \boldsymbol{\omega+1}$, although it has the same support.

Define $X = \mathbb{N} \times Y$ equipped with the lexicographic quasi-ordering $\leq_{\text{lex}}$ defined in Section 5.4. Intuitively, X is the WQO obtained by putting ω copies of Y on top of each other. Elements of X are of the form $\langle i, \langle a, n \rangle \rangle$ where $i \in \mathbb{N}$ designates the copy of Y (the floor of the tower), $a \in \{1, 2\}$ designates the disjoint copy (left or right) and $n \in (\boldsymbol{\omega+1})$. Due to space constraints, only the left copy is labeled on Fig. 8.2). Once again, we extend the quasi-ordering by an equivalence relation E defined by :

- $\langle i, (a, n) \rangle E \langle i, (b, m) \rangle$ for $i \in \mathbb{N}$, $a, b \in \{1, 2\}$ and $n, m \in \mathbb{N}$ and $n, m < t_i$.
- $\langle i, (a, n) \rangle E \langle i, (a, m) \rangle$ for $i \in \mathbb{N}$, $a \in \{1, 2\}$ and $n, m \in \mathbb{N}$ and $n, m \geq t_i$.

Note that each copy of Y has either 3 or 5 equivalence classes (depending on whether T_i halts).

(XR)**:** The representation is described above.

(OD)**:** The quasi-ordering is decidable, as in previous cases by bounded simulations of Turing machines.

(IR)**:** Since the equivalence relation E is "gluing" many elements, there are no infinite strictly increasing sequence of elements within a copy of Y. Thus, the only limit ideal is X itself. All other ideals are principal, hence represented by $\downarrow x$ for $x \in X$.

(ID)**:** Inclusion is then trivially decidable: it is the same as the quasi-ordering (plus the maximal element X).

(PI)**:** Computing principal ideals is also trivial.

(CF)**:** To complement filters:

$\complement \uparrow \langle 0, \langle a, n \rangle \rangle = \emptyset$ when $n \in \mathbb{N}$ and $n < t_0$ and $a \in \{1, 2\}$.

$\complement \uparrow \langle i+1, \langle a, n \rangle \rangle = \downarrow \langle i, \langle 1, \omega \rangle \rangle \cup \downarrow \langle i, \langle 2, \omega \rangle \rangle$ when $n \in \mathbb{N}$ and $n \leq t_{i+1}$ and $a \in \{1, 2\}$.

$\complement \uparrow \langle i, \langle a, n \rangle \rangle = \downarrow \langle i, \langle 3-a, n \rangle \rangle$ when $\omega \geq n \geq t_i$ and $a \in \{1, 2\}$.

Finally, observe that $\downarrow \langle i, \langle 1, \omega \rangle \rangle \cap \downarrow \langle i, \langle 2, \omega \rangle \rangle$ is an ideal if and only if T_i does not halt. Thus, if intersections were computable, the size of the ideal decomposition of the result would decide the halting problem.

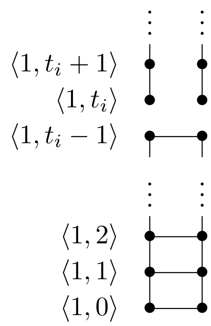

Figure 8.3: X_i when T_i halts. $(X_i, \leq_i / E_i)$ is isomorphic to $\mathbf{1} \oplus (\mathbf{1} \sqcup \mathbf{1})$.

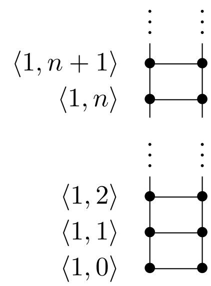

Figure 8.4: X_i when T_i does not halt. $(X_i, \leq_i / E_i)$ is isomorphic to $\mathbf{1}$.

8.2.4 Ideal Decomposition of X (XI)

As mentioned at the beginning of this subsection, the result for (XI) is a little weaker, as for a given X, its ideal decomposition is a constant, hence computable. Thus we here provide an infinite collection of WQOs $(X_i, \leq_i)_{i \in \mathbb{N}}$ such that one cannot compute the ideal decomposition of X_i given $i \in \mathbb{N}$.

Define $X_i = \mathbb{N} \sqcup \mathbb{N}$ with the natural ordering extended with relation E_i defined by (*cf* Figures 8.3 and 8.4):

- $\langle a, n \rangle E_i \langle b, m \rangle$ for $a, b \in \{1, 2\}$ and $n, m < t_i$.
- $\langle a, n \rangle E_i \langle a, m \rangle$ for $a \in \{1, 2\}$ and $n, m \geq t_i$.

First of all, for any $i \in \mathbb{N}$, $(X_i, \leq_i)$ is ideally effective.

(XR): Take the usual representation for disjoint sum (see Chapter 5).

(OD): As in the other cases, the quasi-ordering is decidable by bounded simulations.

(IR): Thanks to the equivalence relation, there are no limit ideals, and we thus represent ideals as elements of X_i.

(ID): Ideal inclusion is then the same as the quasi-ordering on X_i.

(PI): The ideal $\downarrow x$ has the same representation as x, for $x \in X_i$.

(CF): Complements of filters are computed using:

$$\complement \uparrow \langle a, n \rangle = \emptyset \qquad \text{if } n < t_i$$
$$\complement \uparrow \langle a, n \rangle = \downarrow \langle 3 - a, t_i \rangle \qquad \text{if } n \geq t_i$$

(II): Ideal intersections are computed using:

$$\downarrow\langle a,n\rangle \cap \downarrow\langle a,m\rangle = \downarrow\langle a,\min(n,m)\rangle;$$

$$\downarrow\langle a,n\rangle \cap \downarrow\langle 3-a,m\rangle = \begin{cases} \downarrow\langle 1,\min(n,m)\rangle & \text{when } \min(n,m) < t_i \\ \downarrow\langle 1,0\rangle & \text{otherwise.} \end{cases}$$

However, the function that maps $i \in \mathbb{N}$ to the ideal decomposition of X_i is not computable, since otherwise it would decide the halting problem. Indeed, X_i is an ideal if and only if T_i does not halt.

8.3 Minimality of Definition 3.1.1

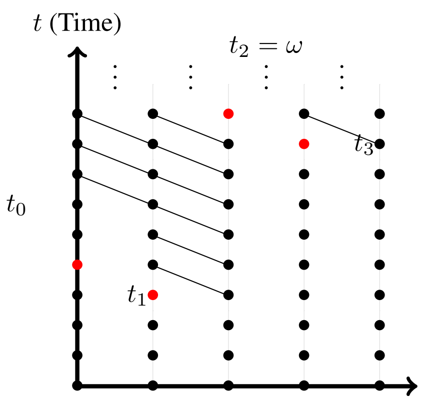

Figure 8.5: Black lines symbolize equivalence relation E. Points used to show that a downward-closed set is not an ideal are highlighted in red.

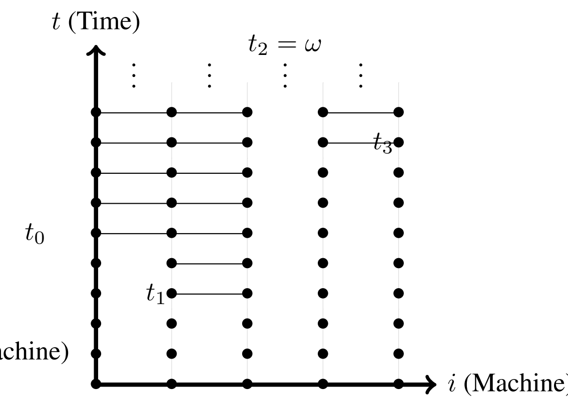

Figure 8.6: Horizontal black lines symbolize equivalence relation E

In the previous sections, we have provided a minimal list of axioms to define ideal effectiveness (Definition 3.1.4). Definition 3.1.1 for effectiveness is not minimal either. For instance, (PI) and (ID) can be used to decide (OD): $x \leq y \iff \downarrow x \in \downarrow y$. However, a WQO is described by $(X, \leq)$, and requiring that $\leq$ is decidable seems natural, eventhough not mandatory.

Besides, removing (XR) or (IR) from the system would not really make sense since they fix the representation for all the other axioms. As noted at the beginning of Section 8.2, we sometimes understand effectiveness and ideal effectiveness as properties verified not only by a WQO, but by a WQO further equipped with representations of its elements and its ideals. Nonetheless, we prove a property which is close to the independence of (IR) from the other axioms: it is always possible to represent downward-closed sets as complements of upward-closed sets, that is, to represent downward-closed sets by excluded minors, as mentioned at the end of the introduction of this part.

We have argued there why we think our representation is better (more symmetry, trivial unions, ...), but here is another drawback of the excluded minor representation: it is undecidable whether a given downward-closed set is an ideal.

Define $X = \mathbb{N}^2$ quasi-ordered by $\leq_\times \circ E$ where E is the smallest equivalence relation such that (Fig. 8.5):

$$\langle i+1, n\rangle E \langle i, n+1\rangle \text{ whenever } n \geq t_i$$

Then, T_i halts if and only if $\complement\uparrow\langle i+1, 0\rangle$ is not an ideal. Indeed, if T_i halts, then $\langle i, t_i\rangle$ and $\langle i-1, t_i+1\rangle$ are in $D = \complement\uparrow\langle i+1, 0\rangle$ but do not have a common upper bound in D. Conversely, if T_i does not halt, then $\complement\uparrow\langle i+1, 0\rangle = \downarrow_\times(i, \omega)$, which is an ideal.

Finally, we formally prove that ideal inclusion (ID) is independent from the other axioms. Define $X = \mathbb{N}^2$ and $\leq = \leq_\times \circ E$ where E is the smallest equivalence relation such that (Fig. 8.6):

$$\langle i, j\rangle E \langle i+1, j\rangle \text{ whenever } j \geq t_i$$

As proved in Section 4.2, the ideals of $(X, \leq)$ are the closure under E of ideals of $(\mathbb{N}^2, \leq_\times)$. Observe that $\downarrow_\times\langle i+1, \omega\rangle \subseteq \overline{\downarrow_\times\langle i, \omega\rangle}$ if and only if T_i halts.

8.4 Deciding whether an Ideal is Principal

We prove that in general one cannot decide whether an ideal is principal, given a full presentation of a WQO and an ideal of this WQO. The counter-example is again built following the same idea. Define $X = \omega^2$, and extend the natural ordering so that for any given $i \in \mathbb{N}$, elements $\omega \cdot i + t$ for $t \geq t_i$ form an equivalence class (Fig. 8.7). We show that $(X, \leq_E)$ is ideally effective using Section 4.2.

- $\mathcal{C}l_{\mathrm{I}}(\boldsymbol{\omega \cdot i + j}) = \boldsymbol{\omega \cdot (i + 1)}$ if $j > t_i$,
- $\mathcal{C}l_{\mathrm{I}}(\boldsymbol{\omega \cdot i + j}) = \boldsymbol{\omega \cdot i + j}$ otherwise.
- $\mathcal{C}l_{\mathrm{F}}(\omega \cdot i + j) = \omega \cdot i + t_i$ if $j \geq t_i$,
- $\mathcal{C}l_{\mathrm{F}}(\omega \cdot i + j) = \omega \cdot i + j$ otherwise.

However, $\boldsymbol{\omega \cdot (i + 1)}$ is a principal ideal if and only if T_i halts, hence, one cannot decide whether an ideal is principal.

This is yet another independence result: one could add this axiom to our definition of ideal effectiveness, and the axiomatic system would still be minimal (Formally, one should also check that the other axiom remain independant in the presence of this one. Observe that for every WQOs considered in Section 8.2, it is decidable whether a given ideal is principal).

Nonetheless, we decided not to include this as an axiom: a priori, deciding whether an ideal is principal does not seem related to our original motivation: handling closed subsets. It turned out to be sometimes related, for instance in Section 5.4 where it is crucial to be able to decide whether an ideal is principal to prove ideal effectiveness of the lexicographic quasi-ordering. This is further discussed in Section 8.4.1. Another case where the unability to test whether an ideal is principal affected our work is discussed in Section 8.4.2. Because of these two examples mostly, we considered adding this requirement to our definition. However, the WQO defined above is obtained as a quotient of an ideally effective WQO, that in addition satisfies the extra assumptions of Section 4.1. This proves that the ability to decide ideal principality is not preserved under quotient. We favored the generality of our results on quotient over the specific lexicographic quasi-ordering, and decided to exclude this axiom from Definition 3.1.4.

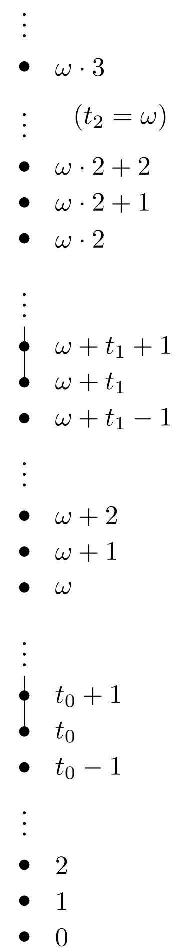

Figure 8.7: WQO for Subsection 8.4

8.4.1 The Lexicographic Quasi-Ordering Is Not Ideally Effective

In this section, we prove that the lexicographic quasi-ordering introduced in Section 5.4 is not ideally effective in general. More precisely, in Section 5.4 we proved that $(A \times B, \leq_{\mathrm{lex}})$ is ideally effective provided that $(A, \leq_A)$ and $(B, \leq_B)$ are ideally effective

WQOs, and that we can decide whether an ideal of A is principal. Subsequently, we justify the necessity of this last assumption.

Let $(A, \leq_A)$ be the WQO defined in the previous section, in particular it is undecidable given $I \in \mathit{Idl}(A)$ whether I is principal. Let B be the finite set $\{a, b\}$ ordered with equality. What matters most here is that B is not an ideal.

Lemma 8.4.1. *Given $I \in \mathit{Idl}(A)$, the downward-closed set $I \times B$ is an ideal of $(A \times B, \leq_{\mathrm{lex}})$ if and only if I is not principal.*

Proof. $(\Rightarrow)$ By contraposition, if $I = \downarrow x$ is a principal ideal for some $x \in A$, then the elements $\langle x, a\rangle$ and $\langle x, b\rangle$ have no common upperbound in $I \times B$.

$(\Leftarrow)$ Let $\langle x, c\rangle, \langle y, d\rangle \in I \times B$. Since I is not principal (but directed), there exists $z \in I$ such that z is *strictly* greater than both x and y. Then $\langle z, a\rangle \in I \times B$, and is greater for $\leq_{\mathrm{lex}}$ than both $\langle x, c\rangle, \langle y, d\rangle$. □

Now given $I \in \mathit{Idl}(A)$, we can compute its complement $U \overset{\mathrm{def}}{=} A \smallsetminus I$. We then compute $V \overset{\mathrm{def}}{=} U \times \{a\} \cup U \times \{b\} \in \mathit{Up}(A \times B)$. Since $I \times B$ is the complement (in $A \times B$) of V, the complement of V cannot be computed, because we would otherwise be able to test whether $I \times B$ is an ideal. This proves that axiom (CF) is not satisfied and $(A \times B, \leq_{\mathrm{lex}})$ is not ideally effective. Note that the above proof does not assume a particular representation of ideals of $(A \times B, \leq_{\mathrm{lex}})$.

8.4.2 The Domination Ordering on Multisets Does Not Effectively Extend the Embedding Quasi-Ordering

Proving $(X^{\circledast}, \leq_{\mathrm{ms}})$ to be ideally effective following the approach of Section 4.1 would require to show that functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ are computable, *i.e.* that downward and upward closures for $\leq_{\mathrm{ms}}$ of an ideal and a filter of $(X^{\circledast}, \leq_{\mathrm{emb}})$ are computable. In this subsection we show that this is not the case in general. This justifies that ideal effectiveness was shown from scratch in Section 7.2, with a better encoding of ideals of $(X^{\circledast}, \leq_{\mathrm{ms}})$ than the one from Section 4.1.

Recall that ideals of $(X^{\circledast}, \leq_{\mathrm{emb}})$ are elements $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$ for $\boldsymbol{B} \in (\mathit{Idl}(X))^{\circledast}$ and $D \in \mathit{Down}(X)$, where $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} = \{M \mid M \smallsetminus D \in_{\mathrm{emb}} \boldsymbol{B}\}$. Observe that when characterizing $\downarrow_{\mathrm{ms}}[\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}]$ in Proposition 7.2.1, it was crucial to distinguish principal ideals from limit ideals. Suppose $\boldsymbol{B} = \{\!|\downarrow x_1, \ldots, \downarrow x_n, I_1, \ldots, I_m|\!\}$ where $I_1, \ldots, I_m$ are limit ordinals, then it was proved in Proposition 7.2.1 that $\downarrow_{\mathrm{ms}}[\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}] = \downarrow_{\mathrm{ms}}\{\!|x_1 \cdots x_n|\!\} + (D \cup I_1 \cup \cdots \cup I_m)^{\circledast}$.

Applying this to the WQO X from Section 8.4:

$$\mathcal{C}l_{\mathrm{I}}(\downarrow_{\in}\{\!|\boldsymbol{\omega \cdot (i+1)}|\!\}) = \begin{cases} \downarrow_{\mathrm{ms}}\{\!|\omega \cdot i + t_i + 1|\!\} & \text{if } T_i \text{ halts} \\ [\boldsymbol{\omega \cdot (i+1)}]^{\circledast} & \text{otherwise} \end{cases}$$
$$= \begin{cases} \downarrow_{\in}\{\!|\boldsymbol{\omega \cdot (i+1)}|\!\} \cup [\boldsymbol{\omega \cdot i + t_i}]^{\circledast} & \text{if } T_i \text{ halts} \\ [\boldsymbol{\omega \cdot (i+1)}]^{\circledast} & \text{otherwise} \end{cases}$$

8.5 Deciding whether an Ideal is Adherent

Finally, we show that adherence cannot be decided in general. Remember from Section 4.3 that given $Y \subseteq X$, an ideal $I \in Idl(X)$ is adherent to Y if and only if $\downarrow_X(I \cap Y) = I$.

Below, we define a WQO $(X, \leq)$ and a recursive subset $Y \subseteq X$ such that deciding whether a given ideal of X is adherent to Y is undecidable. Moreover, we show that this WQO is not ideally effective, proving that some extra assumptions are necessary for induced WQOs to be ideally effective (however, this does not prove that the extra assumptions made in Section 4.3 are necessary).

Define $X = \omega^2$ ordered by the natural ordering. $(X, \leq)$ is ideally effective since it is an ordinal, as proved in Section 3.2.3. Consider the subset $Y = \{\omega \cdot i + t \mid i, t \in \mathbb{N}, t \leq t_i\}$. This is a recursive subset of X Observe that the ideal $\boldsymbol{\omega \cdot (i+1)}$ is adherent to Y if and only if T_i halts. Therefore, adherence to Y is undecidable. Moreover, $Y \smallsetminus \uparrow_Y \omega \cdot (i+1)$ gives the halting time t_i of T_i, hence $(Y, \leq)$ is not ideally effective.

Chapter 9

Toward Ideally Effective BQOs

We motivate our introduction of *Better Quasi-Orderings* with yet another order-construction: that of taking infinite sequences over a WQO $(X, \leq)$. By infinite, we mean sequences of length ω.

9.1 Infinite Sequences of WQOs

We consider the QO $(X^\omega, \leq_\omega)$ where X^ω denotes the set of infinite sequences over X, and $\leq_\omega$ is the embedding relation introduced in Chapter 2 restricted to X^ω. As a reminder, if $\boldsymbol{u} = (u_i)_{i\in\mathbb{N}}$ and $\boldsymbol{v} = (v_i)_{i\in\mathbb{N}}$, then:

$$\boldsymbol{u} \leq_\omega \boldsymbol{v} \overset{\text{def}}{\Leftrightarrow} \text{There exists a strictly increasing mapping } f : \mathbb{N} \to \mathbb{N}.\ \forall i \in \mathbb{N}.\ u_i \leq v_{f(i)}$$

If $|X| > 1$, the set X^ω has uncountably many elements, and thus we cannot represent all its elements algorithmically. However, if $(X, \leq)$ is a WQO, which we now assume, then $(X^\omega, \leq_\omega)$ has countably many equivalence classes for $\equiv_\omega$. This relies on the following characterization of $\leq_\omega$ over X^ω:

Proposition 9.1.1. *Given $\boldsymbol{u} = (u_i)_{i\in\mathbb{N}} \in X^\omega$, define the* tail *of $\boldsymbol{u}$ as $D(\boldsymbol{u}) = \bigcap_{i\in\mathbb{N}} \bigcup_{j\geq i} {\downarrow} u_j$, and the* head *of $\boldsymbol{u}$ as the finite prefix $h(\boldsymbol{u}) = u_0 \cdots u_j$ of $\boldsymbol{u}$, where j is the smallest natural number such that $\forall k > j.\ u_k \in D(\boldsymbol{u})$.*

Then, for all $\boldsymbol{u} \in X^\omega$: ${\downarrow} \boldsymbol{u} = ({\downarrow_*}\, h(\boldsymbol{u})) \cdot D(\boldsymbol{u})^\omega$

Corollary 9.1.2. *Given $\boldsymbol{u}, \boldsymbol{v} \in X^\omega$,*

$$\boldsymbol{u} \leq_\omega \boldsymbol{v} \Leftrightarrow h(\boldsymbol{u}) \in ({\downarrow} h(\boldsymbol{v})) \cdot D(\boldsymbol{v})^* \wedge D(\boldsymbol{u}) \subseteq D(\boldsymbol{v})$$

Proof. Of Proposition 9.1.1 and Corollary 9.1.2.
Let $\boldsymbol{u} = (u_i)_{i\in\mathbb{N}} \in X^\omega$. Intuitively, $D(\boldsymbol{u})$ consists of all elements of X that will be covered infinitely often in $\boldsymbol{u}$. The sequence $D_i = \bigcup_{j\geq i}({\downarrow} u_j)$ is a decreasing sequence of downward-closed subsets of $(X, \leq)$. Thus since $(X, \leq)$ is WQO, $(D_i)_{i\in\mathbb{N}}$ stabilizes to some element D_{i_0}, and its limit is $D(\boldsymbol{u}) = D_{i_0} \neq \emptyset$. Hence, $h(\boldsymbol{u})$ is well defined.

Let $\boldsymbol{u} = (u_i)_{i\in\mathbb{N}}, \boldsymbol{v} = (v_i)_{i\in\mathbb{N}} \in X^\omega$. We show the following:

$$\boldsymbol{u} \in \downarrow \boldsymbol{v} \iff \boldsymbol{u} \in (\downarrow h(\boldsymbol{v})) \cdot D(\boldsymbol{v})^\omega.$$

($\Rightarrow$) Assume $\boldsymbol{u} \leq_\omega \boldsymbol{v}$ and let $f : \mathbb{N} \to \mathbb{N}$ be a witness of this embedding. Let i_0 be the greatest index such that $f(i) \leq |h(\boldsymbol{v})|$. Then $u_1 \cdot u_2 \cdots u_{i_0} \leq_* h(\boldsymbol{v})$ ($\leq_*$ is the embedding quasi-ordering on X^*, see Section 6.1). Moreover, by definition of $h(\boldsymbol{v})$: for every $i > i_0$, $v_{f(i)} \in D(\boldsymbol{v})$. Thus $u_i \in D(\boldsymbol{v})$.

($\Leftarrow$) Assume $\boldsymbol{u} \in (\downarrow h(\boldsymbol{v})) \cdot D(\boldsymbol{v})^\omega$. Let $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{u}_2$ with $\boldsymbol{u}_1 \in \downarrow h(\boldsymbol{v})$ and $\boldsymbol{u}_2 \in D(\boldsymbol{v})^\omega$. We define $f : \mathbb{N} \to \mathbb{N}$ that witnesses $\boldsymbol{u} \leq_\omega \boldsymbol{v}$. For indexes up to $|\boldsymbol{u}_1|$, f is defined as a witness of $\boldsymbol{u}_1 \leq_* h(\boldsymbol{v})$. The remainder of f is defined inductively: for every $i > |\boldsymbol{u}_1|$, pick for $f(i)$ an index j such that $j > \max\{i, f(1), f(2), \ldots, f(i-1)\}$ and $v_j \geq u_i$. Such an index j exists since $u_i \in D(\boldsymbol{v})$ and $D(\boldsymbol{v})$ consists of all the elements of X that are covered infinitely often in $\boldsymbol{v}$, that is: $x \in D(\boldsymbol{v}) \Leftrightarrow \forall i \in \mathbb{N}.\ \exists j > i.\ x \leq v_j$.

Proof of Corollary 9.1.2

$$\begin{aligned}
\boldsymbol{u} \leq_\omega \boldsymbol{v} &\Leftrightarrow \downarrow \boldsymbol{u} \subseteq \downarrow \boldsymbol{v} \\
&\Leftrightarrow (\downarrow h(\boldsymbol{u})) \cdot D(\boldsymbol{u})^\omega \subseteq (\downarrow h(\boldsymbol{v})) \cdot D(\boldsymbol{v})^\omega \\
&\Leftrightarrow (\downarrow h(\boldsymbol{u})) \cdot D(\boldsymbol{u})^\omega \subseteq (\downarrow h(\boldsymbol{v})) \cdot D(\boldsymbol{v})^* \cdot D(\boldsymbol{v})^\omega \\
&\Leftrightarrow h(\boldsymbol{u}) \in \downarrow h(\boldsymbol{v}) \cdot D(\boldsymbol{v})^* \wedge D(\boldsymbol{u}) \subseteq D(\boldsymbol{v})
\end{aligned}$$

□

Corollary 9.1.3. *The set $X^\omega / \equiv_\omega$ is countable.*

Proof. It follows from Proposition 9.1.1 that

$$u \not\equiv_\omega v \Leftrightarrow h(\boldsymbol{u}) \neq h(\boldsymbol{v}) \vee D(\boldsymbol{u}) \neq D(\boldsymbol{v})$$

Since X^* and $Down(X)$ are countable (X is assumed countable), so is $X^\omega / \equiv_\omega$. □

Finally, one last property is directly implied by Proposition 9.1.1:

Corollary 9.1.4. *$(X^\omega, \leq_\omega)$ is a WQO if and only if $(Down(X), \subseteq)$ is.*

Note that (WQO7) in Section 2.3 only gives well-foundedness of $(Down(X), \subseteq)$. And indeed, $(Down(X), \subseteq)$ may not be a WQO. Hence, neither is $(X^\omega, \leq_\omega)$. This matter is discussed in Section 9.2, and for the remainder of this chapter, we now assume that both $(X, \leq)$ and $(Down(X), \subseteq)$ are WQOs. As a consequence, $(X^\omega, \leq_\omega)$ is a WQO as well.

Coming back to the representation of elements of X^ω (XR), we have argued that it is hopeless to represent every element of X^ω, and we will therefore focus on the ideal effectiveness of its quotient $(X^\omega / \equiv_\omega, \leq_\omega)$. The above results suggest to represent equivalence class of $X^\omega / \equiv_\omega$ as pairs from $X^* \times Down(X)$. Intuitively, we want a pair $\langle \boldsymbol{u}, D\rangle \in X^* \times Down(X)$ to encode the equivalence class of infinite sequences that have head $\boldsymbol{u}$ and tail D. There are three issues with this representation:

1. If $D = \emptyset$, there are no infinite sequences with tail D. We therefore define $Down'(X) = Down(X) \setminus \{\emptyset\}$, the set of non-empty downward-closed subsets of X. Note that working with $X^* \times Down(X)$ amounts to represent $X^{\leq\omega}$, the set of sequences of length at most ω. Subsequently, we will restrict our attention to $X^* \times Down'(X)$.

2. If $\boldsymbol{u}$ has a non-empty suffix in D, then there are no infinite sequence of head $\boldsymbol{u}$ and tail D, since we defined the head to be as small as possible. We therefore want $\langle \boldsymbol{u}, D\rangle$ to denote any infinite sequence in $\boldsymbol{u} \cdot D^\omega$.

3. However, all the elements of $\boldsymbol{u} \cdot D^\omega$ do not belong to the same equivalence class. For instance, if $X = \{a, b, c\}$ is a finite alphabet, and if $\boldsymbol{u} = aaba$ and $D = \{a, c\}$, then $aaba(a + c)^\omega$ contains the word $aabaa^\omega$, which is strictly smaller than $aaba(ca)^\omega$ for instance. The reason is that $D(aabaa^\omega) = \{a\} \subsetneq D$. The solution is to restrict our attention to the maximal elements of $\boldsymbol{u} \cdot D^\omega$: they are all equivalent, and thus represent a unique element of $X^\omega / \equiv_\omega$. This is proved in the next proposition.

Proposition 9.1.5. *1. Let $\boldsymbol{u}, \boldsymbol{v} \in X^\omega$. If $\boldsymbol{u} \equiv_\omega \boldsymbol{v}$, then $h(\boldsymbol{u}) = h(\boldsymbol{v})$ and $D(\boldsymbol{u}) = D(\boldsymbol{v})$. We can thus define the* head *and* tail *of an equivalence class $\boldsymbol{S} \in X^\omega / \equiv_\omega$. They will be denoted $h(\boldsymbol{S})$ and $D(\boldsymbol{S})$ respectively.*

2. Given an equivalence class $\boldsymbol{S}$ of $X^\omega / \equiv_\omega$, $\boldsymbol{S}$ consists exactly of the maximal elements of $h(\boldsymbol{S}) \cdot (D(\boldsymbol{S}))^\omega$.

3. Given $\langle \boldsymbol{u}, D\rangle \in X^* \times Down'(X)$, the maximal elements of $\boldsymbol{u} \cdot D^\omega$ all are equivalent for $\equiv_\omega$. Let $\boldsymbol{S}$ be the equivalence class they all belong to. We have $D(\boldsymbol{S}) = D$ and $h(\boldsymbol{S}) = \boldsymbol{u}_1$, where $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{u}_2$ and $\boldsymbol{u}_2$ is the longest suffix of $\boldsymbol{u}$ which is in D^*.

According to the third point of the proposition above, we let $\langle \boldsymbol{u}, D\rangle \in X^* \times Down'(X)$ represent the equivalence class $\boldsymbol{S} \in X^\omega / \equiv_\omega$ to which every maximal element of $\boldsymbol{u} \cdot D^\omega$ belongs. Note that in general the maximal elements of $\boldsymbol{u} \cdot D^\omega$ are strictly included in $\boldsymbol{S}$. For instance, in the case introduced before: if $X = \{a, b, c\}$ is a finite alphabet, and if $\boldsymbol{u} = aaba$ and $D = \{a, c\}$, the infinite sequence $aab(ca)^\omega$ is equivalent to maximal elements of $\boldsymbol{u} \cdot D^\omega$, but does not belong to it.

Moreover, an equivalence class $\boldsymbol{S}$ has several representations. In light of the proposition, we define $\langle h(\boldsymbol{S}), D(\boldsymbol{S})\rangle$ to be its *canonical representation*. According to the third point, this representation is computable from any other representation $\langle \boldsymbol{u}, D\rangle$: it suffices to remove the suffix of $\boldsymbol{u}$ which is in D.

Proof. Of Proposition 9.1.5

1. According to Corollary 9.1.2, we immediately get that $D(\boldsymbol{u}) = D(\boldsymbol{v})$ by double inclusion. Moreover, we have $h(\boldsymbol{u}) \in \downarrow h(\boldsymbol{v}) \cdot D(\boldsymbol{u})^*$, but by definition of $h(\boldsymbol{u})$, its last symbol is not in $D(\boldsymbol{u})$. Therefore $h(\boldsymbol{u}) \leq_* h(\boldsymbol{v})$. By symmetry we get $h(\boldsymbol{u}) = h(\boldsymbol{v})$.

2. Any element $\boldsymbol{u} \in X^\omega$ belongs to $h(\boldsymbol{u}) \cdot D(\boldsymbol{u})^\omega$. Therefore, elements of $\boldsymbol{S}$ belong to $h(\boldsymbol{S}) \cdot D(\boldsymbol{S})^\omega$. Conversely, if $\boldsymbol{u} \in h(\boldsymbol{S}) \cdot D(\boldsymbol{S})^\omega$ then $h(\boldsymbol{u}) \in (\downarrow h(\boldsymbol{S})) \cdot D(\boldsymbol{S})^*$ and $D(\boldsymbol{u}) \subseteq D(\boldsymbol{S})$. Therefore, by Corollary 9.1.2, $\boldsymbol{u}$ is smaller than any element of $\boldsymbol{S}$.

3. Given $\boldsymbol{u} \in X^*$ and $D \in \mathit{Down}'(X)$, decompose $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{u}_2$ where $\boldsymbol{u}_2$ is the longest suffix of $\boldsymbol{u}$ which is in D. Any maximal word $\boldsymbol{v}$ of $\boldsymbol{u} \cdot D^\omega$ can be written $\boldsymbol{u}_1\boldsymbol{v}'$ with $\boldsymbol{v} \in D^\omega$. Since $\boldsymbol{u}_1$'s last element is not in D, $h(\boldsymbol{v}) = \boldsymbol{u}_1$ and $D(\boldsymbol{v}) = D$. Therefore, they are all equivalent, and we denote by $\boldsymbol{S}$ their equivalence class. We have proved that $\boldsymbol{S}$ is the class of elements of head $\boldsymbol{u}_1$ and tail D.

□

The decidability of $\leq_\omega$ on this representation follows from Corollary 9.1.2, and the computability of the canonical representation, the quasi-ordering $\leq$ on X, the inclusion of ideals of $(X^*, \leq_*)$ and the inclusion of downward-closed subsets of X.

We denote by the same symbol $\leq_\omega$ the quasi-ordering on $X^* \times \mathit{Down}'(X)$ defined by: $\langle \boldsymbol{u}, D\rangle \leq_\omega \langle \boldsymbol{v}, E\rangle$ if all the equivalent maximal sequences of $\boldsymbol{u} \cdot D^\omega$ are smaller than all the equivalent maximal sequences of $\boldsymbol{v} \cdot E^\omega$. This new quasi-ordering $\leq_\omega$ is decidable since we can always compute the canonical representations, and then use Corollary 9.1.2. Instead, we prove that Corollary 9.1.2 is also valid in this more general setting:

Proposition 9.1.6. *Given two pairs $\langle \boldsymbol{u}, D\rangle, \langle \boldsymbol{v}, E\rangle \in X^* \times \mathit{Down}'(X)$,*

$$\langle \boldsymbol{u}, D\rangle \leq_\omega \langle \boldsymbol{v}, E\rangle \iff \boldsymbol{u} \in (\downarrow \boldsymbol{v}) \cdot E^* \wedge D \subseteq E$$

Proof. Let $\boldsymbol{S}$ and $\boldsymbol{T}$ be the two equivalence classes of $X^\omega/\equiv_\omega$ represented by $\langle \boldsymbol{u}, D\rangle$ and $\langle \boldsymbol{v}, E\rangle$ respectively. By Proposition 9.1.5, we know that $D = D(\boldsymbol{S})$ and $E = D(\boldsymbol{T})$ on the one hand, and that $\boldsymbol{u} = h(\boldsymbol{S}) \cdot \boldsymbol{u}'$ and $\boldsymbol{v} = h(\boldsymbol{T}) \cdot \boldsymbol{v}'$ for some $\boldsymbol{u}' \in D^*$ and some $\boldsymbol{v}' \in E^*$.

($\Rightarrow$) If $\langle \boldsymbol{u}, D\rangle \leq_\omega \langle \boldsymbol{v}, E\rangle$, by Corollary 9.1.2:

$$h(\boldsymbol{S}) \in (\downarrow h(\boldsymbol{T})) \cdot D(\boldsymbol{T})^* \wedge D(\boldsymbol{S}) \subseteq D(\boldsymbol{T}),$$

which is equivalent to $h(\boldsymbol{S}) \in (\downarrow h(\boldsymbol{T})) \cdot E^* \wedge D \subseteq E$. The left conjunct implies $h(\boldsymbol{S})\boldsymbol{u}' \in (\downarrow h(\boldsymbol{T})) \cdot (\downarrow \boldsymbol{u}') \cdot E^*$, and since $\boldsymbol{u}' \in D^* \subseteq E^*$, this simplifies to $\boldsymbol{u} \in (\downarrow h(\boldsymbol{T})) \cdot E^* \subseteq (\downarrow \boldsymbol{v}) \cdot E^*$.

($\Leftarrow$) Conversely, because $\boldsymbol{u} = h(\boldsymbol{S})\boldsymbol{u}'$ and $\boldsymbol{u}' \in D^* \subseteq E^*$, $\boldsymbol{u} \in (\downarrow \boldsymbol{v}) \cdot E^*$ implies $h(\boldsymbol{S}) \in (\downarrow h(\boldsymbol{T})\boldsymbol{v}') \cdot E^*$. But since $\boldsymbol{v}' \in E^*$, the right-hand-side simplifies to $(\downarrow h(\boldsymbol{T})) \cdot E^*$, entailing the desired condition:

$$h(\boldsymbol{S}) \in (\downarrow h(\boldsymbol{T})) \cdot D(\boldsymbol{T})^* \wedge D(\boldsymbol{S}) \subseteq D(\boldsymbol{T})$$

□

It follows from the previous proposition that the relation $\leq_\omega$ on $X^* \times \mathit{Down}'(X)$ extends the classical product ordering $\leq_\times$ (it is actually be obtained as a composition with an equivalence relation, as defined in Section 4.2). Therefore, Section 4.1 applies to the quasi-ordering $\leq_\omega$, and we therefore once again turn to functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$.

Theorem 9.1.7. *Given $\boldsymbol{I} \in \mathit{Idl}(X^*, \leq_*)$, $\mathcal{J} \in \mathit{Idl}(\mathit{Down}(X), \subseteq)$, $\boldsymbol{u} \in X^*$ and $D \in \mathit{Down}(X)$:*

$$\mathcal{C}l_{\mathrm{I}}(\boldsymbol{I} \times \mathcal{J}) \stackrel{\mathit{def}}{=} {\downarrow_\omega}(\boldsymbol{I} \times \mathcal{J}) = (\boldsymbol{I} \cdot (\bigcup \mathcal{J})^*) \times \mathcal{J}$$

$$\mathcal{C}l_{\mathrm{F}}(\langle \boldsymbol{u}, D\rangle) \stackrel{\mathit{def}}{=} {\uparrow_\omega}\langle \boldsymbol{u}, D\rangle = \bigcup_{\boldsymbol{u}=\boldsymbol{u}_1\boldsymbol{u}_2} {\uparrow_\times}\langle \boldsymbol{u}_1, D \cup {\downarrow} \mathit{Supp}(\boldsymbol{u}_2)\rangle$$

Proof. Let $\langle \boldsymbol{u}, D\rangle \in \boldsymbol{I} \times \mathcal{J}$ and $\langle \boldsymbol{v}, E\rangle \leq_\omega \langle \boldsymbol{u}, D\rangle$, that is $\boldsymbol{v} \in ({\downarrow} \boldsymbol{u}) \cdot D^*$ and $E \subseteq D$. Since $D \in \mathcal{J}$ and $\boldsymbol{u} \in \boldsymbol{I}$ which is downward-closed, $\boldsymbol{v} \in \boldsymbol{I} \cdot (\bigcup \mathcal{J})^*$. Finally, since $\mathcal{J}$ is downward-closed, $E \in \mathcal{J}$ and $\langle \boldsymbol{v}, E\rangle \in I \cdot (\bigcup \mathcal{J})^*) \times \mathcal{J}$.

Conversely, given $\langle \boldsymbol{v}, E\rangle \in \boldsymbol{I} \cdot (\bigcup \mathcal{J})^*) \times \mathcal{J}$, decompose $\boldsymbol{v} = \boldsymbol{u}\boldsymbol{w}$ with $\boldsymbol{u} \in \boldsymbol{I}$ and $\boldsymbol{w} \in (\bigcup \mathcal{J})^*$. Since $\boldsymbol{w}$ is finite, it actually belongs to a finite union $(\bigcup_i D_i)^*$ of downward-closed sets of $\mathcal{J}$. Since $\mathcal{J}$ is directed, $D \stackrel{\mathrm{def}}{=} E \cup \bigcup_i D_i$ is in $\mathcal{J}$. Therefore, $\langle \boldsymbol{v}, E\rangle \leq_\omega \langle \boldsymbol{u}, D\rangle \in \boldsymbol{I} \times \mathcal{J}$.

Let $\langle \boldsymbol{v}, E\rangle \geq_\omega \langle \boldsymbol{u}, D\rangle$, that is $\boldsymbol{u} \in ({\downarrow_*} \boldsymbol{v}) \cdot E^*$ and $D \subseteq E$. Decompose $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{u}_2$ with $\boldsymbol{u}_1 \leq_* \boldsymbol{v}$ and $\boldsymbol{u}_2 \in E^*$: $\langle \boldsymbol{u}_1, D \cup {\downarrow} \mathit{Supp}(\boldsymbol{u}_2)\rangle \leq_\times \langle \boldsymbol{v}, E\rangle$.

Conversely, if $\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{u}_2$, then trivially $\boldsymbol{u} \in ({\downarrow_*} \boldsymbol{u}_1) \cdot ({\downarrow} \mathit{Supp}(\boldsymbol{u}_2))^*$ and $D \subseteq D \cup {\downarrow} \mathit{Supp}(\boldsymbol{u}_2)$. Thus, according to Corollary 9.1.2, $\langle \boldsymbol{u}_1, D \cup \mathit{Supp}(\boldsymbol{u}_2)\rangle \in {\uparrow_\omega}\langle \boldsymbol{u}, D\rangle$. □

Now according to Sections 4.1, 5.3 and 6.1, $(X^\omega, \leq_\omega)$ is ideally effective provided that:

1. $(X, \leq)$ is ideally effective,
2. $(\mathit{Down}'(X), \subseteq)$ is ideally effective,
3. Functions $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ are computable.

The first condition is the basic assumption we have made throughout other sections. The second one however is novel: indeed, $(\mathit{Down}(X), \subseteq)$ might not even be a WQO in general, making its ideal effectiveness a non-relevant question. The necessity of this extra assumption will be further discussed in the next section. Finally, we could neither prove that the third condition follows from the two first conditions, nor that it was independant. The two constructs we use in Theorem 9.1.7 whose computability does not follow *a priori* from the two first conditions listed above are $\bigcup \mathcal{J}$ (used in the expression of $\mathcal{C}l_{\mathrm{I}}$) and ${\downarrow} \mathit{Supp}(\boldsymbol{u})$ (used in the expression of $\mathcal{C}l_{\mathrm{F}}$). We have already used the second one earlier, and it seems obviously computable, since $\mathit{Supp}(\boldsymbol{u})$ (for $\boldsymbol{u}$ a finite word) is a finite set, ${\downarrow} \mathit{Supp}(\boldsymbol{u}) = \bigcup_{x \in \mathit{Supp}(\boldsymbol{u})} {\downarrow} x$. However, this expression will output a downward-closed set represented using axiom (IR) (representability of ideals of X). But, we need this downward-closed set to be represented using axiom (XR) for $\mathit{Down}'(X)$ (representability of elements of $\mathit{Down}'(X)$). Indeed, if we only assume that both $(X, \leq)$ and $(\mathit{Down}'(X), \subseteq)$ are ideally effective, nothing ensures that the representation of elements of $\mathit{Down}'(X)$ is the same as the representation for downward-closed sets of X (that is, finite sets of ideals of X). We solve this issue by further assuming the computability of the *transfer* function:

$$\begin{cases} Down(X) & \to & Down'(X) \\ D & \mapsto & D \end{cases}$$

Similarly for the construct $\bigcup \mathcal{J}$, it is natural to link the representations for X and $Down'(X)$ in the other direction. We thus assume the computability of the *flattening* function:

$$\begin{cases} Idl(Down'(X)) & \to & Down(X) \\ \mathcal{J} & \mapsto & \bigcup \mathcal{J} \end{cases}$$

To sum up, if $(X, \leq)$ and $Down'(X), \subseteq)$ both are ideally effective WQOs, and if the transfer and flattening functions defined above are computable, then $(X^\omega, \leq_\omega)$ is an ideally effective WQO. As stated previously, the next chapter is devoted to a deeper understanding of the assumptions that $(Down(X), \subseteq)$ is a WQO, and an ideally effective one.

Remark 9.1.8. *It is natural to assume that we represent the elements of $Down'(X)$ using the same representation we have used so far, thus rendering the transfer function trivial. Note that in the expression of $\mathcal{C}l_{\mathrm{F}}$, we also perform unions of downward-closed sets, which is trivial with our usual representation, but might not be for an arbitrary one. Thanksfully, union of downward-closed sets corresponds to intersection of filters:*

$$\uparrow D_1 \cap \uparrow D_2 = \uparrow (D_1 \cup D_2)$$

Unions of downward-closed sets are therefore computable when $Down'(X)$ is ideally effective.

However, even if we assume the usual representation for $Down'(X)$, we could not prove that the computability of the flattening function is implied by the other assumptions (namely ideal effectiveness of both X and $Down(X)$). Nor could we prove that it is independant (in the fashion of Chapter 8).

However, we would like to point out that the flattening function allows to test whether an ideal is principal: $\mathcal{J} \in Idl(Down(X))$ is principal if and only if $\bigcup \mathcal{J} \in \mathcal{J}$. This does not suffice to prove that the flattening function is not computable in general since the counter-example for deciding whether an ideal is principal given in Section 8.4 takes place in a more general setting. In other words, the flattening function allows to decide whether an ideal is principal only in WQOs that are the ideal space of another WQO.

Furthermore, assume we repreent downward-closed sets as finite sets of ideals. Then, the flattening allows to decide whether an ideal of X is principal. Indeed, given an ideal of X, one can compute $\downarrow_{\subsetneq} I \stackrel{def}{=} \{J \in Idl(X) \mid J \subsetneq I\}$. Indeed, this set can be obtained as $\downarrow_{Idl(X)} I \cap (Idl(X) \smallsetminus \uparrow_{Idl(X)} I)$. We claim that $\bigcup(\downarrow_{\subsetneq} I) = I$ if and only if I is not principal in $Idl(X)$. Indeed, if I is principal, $I = \downarrow x$ for some $x \in X$. Then for every $J \in \downarrow_{\subsetneq} I$, $x \notin J$, and therefore $x \notin \bigcup \downarrow_{\subsetneq} I$, but $x \in I$. Conversely, if $x \in I$ but $x \notin \bigcup(\downarrow_{\subsetneq} I)$, then $\downarrow x \subseteq I$ but $\downarrow x \not\subsetneq I$, i.e. *$\downarrow x = I$.*

Once again, this is not enough to conclude that the flattening function is not computable in general: it might be the case the the assumption $Down'(X)$ is ideally effective implies that we can test whether an ideal of X is principal. In particular, for *all ideally effective WQOs $(X, \leq)$ we know of such that we cannot decide whether an ideal is principal, $Down'(X)$ is not ideally effective.*

9.2 Better Quasi-Orderings

Chapters 5 to 7 provide an effective algebra of ideally effective WQOs: any WQO obtained as sums of products of finite sequences of finite subsets of finite multisets of ordinals and/or finite WQOs is an ideally effective WQO. And procedures for set-theoretic operations in this WQO do not only exist, they can be computed from the structure of the WQO.

However, the operations of taking the infinite sequences of a WQO (quasi-ordered with embedding), or the infinite powerset (quasi-ordered with $\sqsubseteq_{\mathcal{H}}$) are not *a priori* part of this algebra. Indeed, these two operations do not preserve the WQO property. This was first proved by Rado in [38]: he exhibited a WQO $(X, \leq)$ such that $(X^\omega, \leq_\omega)$ is not WQO. This WQO is now commonly known as *Rado's structure*. Moreover, this counter example is minimal: given a WQO $(X, \leq)$, $(X^\omega, \leq_\omega)$ is WQO if and only if $(X, \leq)$ does not embed Rado's structure.

Inspired by this characterization, Nash-Williams [39, 40] defined the notion of *Better Quasi-Orderings* (BQO), that notably satisfies the following property: if $(X, \leq)$ is a BQO, then the ordinal sequences over X, quasi-ordered with embeddability (the obvious generalization of $\leq_\omega$ introduced in the previous chapter) is a WQO. Alternatively, Section 9.1 has shown that $\mathcal{P}(X)$ is WQO if and only if X^ω is, and thus, Rado provided a sufficient and necessary condition for $\mathcal{P}(X)$ to be WQO. What Nash-Williams intended are conditions on $(X, \leq)$ such that $\mathcal{P}(X)$, $\mathcal{P}^2(X) \stackrel{\text{def}}{=} \mathcal{P}(\mathcal{P}(X))$, $\mathcal{P}^3(X)$, and so on iterating transfinitely all are WQOs.

Many positive results on BQOs followed, the intuitive rule being that whenever an order-construction preserves WQO, its infinitary version preserves BQO: if $(X, \leq)$ is WQO, then the set of finite sequences over X is WQO (Higman's Theorem), while if it is BQO, the set of sequences over a countable ordinal is BQO. Similar results for sets and trees have been proved.

Of course, BQOs are *better* WQOs, that is a BQO is in particular a WQO, and indeed the original definition was given in a similar way: $(X, \leq)$ is BQO if any application $f : B \to X$ from a barrier B to X is good, that is there exists $s \triangleleft t \in B$ such that $f(s) \leq f(t)$. The formal definitions of barriers and $\triangleleft$ are too technical and out of the scope of this thesis, and will therefore not be given, we simply stress the similarities with (WQO1): the application $f : B \to X$ is a generalization of an infinite sequence (application from $\mathbb{N}$ to X). For that matter, $\mathbb{N}$ is a barrier on which $\triangleleft$ coincides with the natural ordering, which proves that any BQO is WQO.

Jullien [41] later provided an alternative definition of BQOs, which is proved equivalent in [42, 11]. This definition uses the notion of indecomposability of an ordinal sequence (which we solely call sequence from now on, see Chapter 2 for definitions, terminology and notations. In particular, in what follows, $\leq$ denotes sequence embedding). A sequence $\boldsymbol{s}$ is said to be *indecomposable* if for any sequences $\boldsymbol{s}_1, \boldsymbol{s}_2$ such that $\boldsymbol{s} = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$ and $\boldsymbol{s}_2 \neq \epsilon$, $\boldsymbol{s} \leq \boldsymbol{s}_2$, that is indecomposable sequences embed in all their non-empty suffixes. Note that an indecomposable sequence necessarily has an indecomposable length, where an ordinal α is *indecomposable* if it cannot be written $\alpha = \beta + \gamma$ with $0 \neq \gamma < \alpha$, or equivalently, $\alpha = \omega^\beta$ for some ordinal β.

Definition 9.2.1. *Jullien [41] A QO $(X, \leq)$ is BQO if any non-empty sequence $\boldsymbol{s}$ over X of countable length can be written $\boldsymbol{s} = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$ with $\boldsymbol{s}_2$ non-empty and indecomposable,* i.e. *$\boldsymbol{s}$ has an indecomposable suffix.*

In this chapter, we are looking for sufficient conditions on $(X, \leq)$ to ensure that $(X^\omega, \leq_\omega)$ is an ideally effective WQO. In particular, we need $(Down(X), \subseteq)$ to be WQO (*cf.* Corollary 9.1.4). Note that since every construction presented in the previous chapters that preserve WQOs also preserve BQOs, and since our basic WQOs (namely natural numbers and finite quasi-orderings) are BQOs, our effective algebra of WQOs is actually an algebra of BQOs. Thus, we can also close this algebra under the construction of the infinite sequences (which preserves BQO), or the full powerset construction. But what about effectiveness ? To prove that $(X^\omega, \leq_\omega)$ is an ideally effective WQO, we need that $(Down(X), \subseteq)$ is not only a WQO, but an ideally effective one. In other words, we want to show that $Down$ is an ideally effective construction on BQOs. It cannot be an ideally effective construction on WQOs since it does not even preserve WQOs. In Section 9.4, we will actually show a stronger statement. But first, we introduce a finer notion than BQO.

9.3 α-WQO

Observe that Definition 9.2.1 can be naturally layered: X is a BQO if *all* ordinal sequences over X satisfy some property. In the following definition, we only ask that this property holds for ordinal sequences up to some length. The following definition can be found in [11]:

Definition 9.3.1. *[11], Chapter 8*
Given an indecomposable ordinal α, a QO $(X, \leq)$ is α-WQO if any β-sequence $\boldsymbol{s}$ over X for $\beta \leq \alpha$ can be written $\boldsymbol{s} = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$ with $\boldsymbol{s}_2$ non-empty and indecomposable.

Note that the notion of α-WQO when α is not indecomposable coincides with the notion β-WQO if there exists γ such that $\alpha = \gamma + \beta$. Indeed, if $(X, \leq)$ is β-WQO, then any sequence $\boldsymbol{s} \in X^\alpha$ can be decomposed $\boldsymbol{s} = \boldsymbol{uv}$ with $\boldsymbol{v} \in X^\beta$, and thus $\boldsymbol{v}$ has an indecomposable suffix. This is the reason we are only interested in α-WQOs for indecomposable ordinals α.

With this definition, a QO is BQO if and only if it is α-QO for every countable ordinal α. Moreover, the notion of WQO coincides with ω-WQO, which provides a simple proof that BQOs are WQOs.

Proposition 9.3.2. *$(X, \leq)$ is WQO if and only if it is ω-WQO.*

Proof. Observe that ω-sequences have been introduced in the previous chapter.

($\Rightarrow$) If $(X, \leq)$ is WQO, and $\boldsymbol{s}$ is an ω-sequence, then the decomposition of $\boldsymbol{s}$ is given by its head and tail which were introduced in Proposition 9.1.1. Indeed, let $\boldsymbol{s}'$ be the ω-sequence such that $\boldsymbol{s} = h(\boldsymbol{s}) \cdot \boldsymbol{s}'$. It is not hard to see that $\boldsymbol{s}'$ is indecomposable since it is made of elements that are covered infinitely often in $\boldsymbol{s}$.

($\Leftarrow$) If $(X, \leq)$ is ω-WQO, and $\boldsymbol{s} \in X^\omega$, let $\boldsymbol{s} = \boldsymbol{s}_1\boldsymbol{s}_2$ be the decomposition given by Definition 9.3.1, and let x be its first element. Since $\boldsymbol{s}_2$ is indecomposable, x must be smaller than some subsequent element of $\boldsymbol{s}_2$.

$\square$

We are mainly interested in this finer definition for the following property: $(Down(X), \subseteq)$ is WQO if and only if $(X, \leq)$ is ω^2-WQO. This is proved in the following proposition, among other characterizations:

Proposition 9.3.3. *The following are equivalent:*

1. $(X, \leq)$ *is* ω^2*-WQO,*
2. $(Idl(X), \subseteq)$ *is WQO,*
3. $(Down(X), \subseteq)$ *is WQO,*
4. $(Up(X), \supseteq)$ *is WQO,*
5. $(\mathcal{P}(X), \sqsubseteq_\mathcal{H})$ *is WQO,*
6. $(\mathcal{P}(X), \sqsubseteq_\mathcal{S})$ *is WQO,*
7. $(\mathcal{P}_f(X), \sqsubseteq_\mathcal{S})$ *is WQO,*
8. $(X^\omega, \leq_\omega)$ *is WQO,*
9. $X^{<\omega^2}$ *is WQO for the sequence embedding quasi-ordering,*
10. X^{ω^2} *is well-founded for the sequence embedding quasi-ordering,*
11. $(X, \leq)$ *is a WQO and does not contain Rado's structure,*

where $\sqsubseteq_\mathcal{S}$ *is the Smyth quasi-ordering on* $\mathcal{P}(X)$*, defined by:* $S \sqsubseteq_\mathcal{S} T \overset{def}{\Leftrightarrow} \forall y \in T.\, \exists x \in S.\, x \leq y$*.*

Proof. 2 $\iff$ 3: follows from the isomorphism between $(Down(X), \subseteq)$ and $(\mathcal{P}_f(Idl(X))/{\equiv_\mathcal{H}}, \sqsubseteq_\mathcal{H})$, and the fact that $(Y, \leq)$ is WQO if and only if $(\mathcal{P}_f(Y), \sqsubseteq_\mathcal{H})$ is.

5 $\iff$ 3 $\iff$ 4: also follows from aforementioned isomorphisms (notably at the end of Chapter 2).

4 $\iff$ 6 $\iff$ 7: dual of $\sqsubseteq_\mathcal{H}$ which coincides with inclusion for downward-closed sets, $\sqsubseteq_\mathcal{S}$ coincides with $\supseteq$ for upward-closed set: $S \sqsubseteq_\mathcal{S} T$ if and only if $\uparrow S \supseteq \uparrow T$. Besides, any subset S is equivalent for $\sqsubseteq_\mathcal{S}$ to its upward-closure, which is equivalent to its finite basis.

3 $\iff$ 8: follows from Corollary 9.1.4

9 $\Rightarrow$ 8: X^ω is a subset of $X^{<\omega^2}$.

8 $\Rightarrow$ 9: The embedding quasi-ordering on $X^{<\omega^2}$ is an extension of Higman's quasi-ordering on $(X^\omega)^*$, which is a WQO by Higman's Lemma.

10 $\Rightarrow$ 1: Let $\boldsymbol{s} \in X^{\omega^2}$. Consider the set of all non-empty suffixes of $\boldsymbol{s}$. Since X^{ω^2} is well-founded, this set has a minimal element. It is simple to see that this minimal element is an indecomposable suffix of $\boldsymbol{s}$.

$1 \Rightarrow 8$: Let Y be the set of indecomposable sequences of X^ω. We show that Y is a WQO. It follows that $X^* \times Y$ is a WQO, since by Proposition 9.3.2, $(X, \leq)$ is WQO. Moreover, since $(X, \leq)$ is ω^2-WQO, any ω-sequence $\boldsymbol{s}$ can be decomposed $\boldsymbol{s} = \boldsymbol{s}_1\boldsymbol{s}_2$ with $\boldsymbol{s}_2$ indecomposable. Therefore, X^ω is isomorphic to an extension of $X^* \times Y$, and as such is WQO.

We proceed to show that Y is WQO. An infinite sequence $\boldsymbol{S} = (\boldsymbol{s}_i)_{i<\omega}$ of elements of Y can be *flattened* as a sequence $\boldsymbol{s} = \prod_{i<\omega} \boldsymbol{s}_i \in X^{\omega^2}$. This sequence can be written $\boldsymbol{s} = \boldsymbol{uv}$ with $\boldsymbol{v}$ indecomposable, and we can further assume that the length of $\boldsymbol{u}$ is a multiple of ω, that is $\boldsymbol{u} = \boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_n$ for some $n < \omega$. Now we can *unflatten* $\boldsymbol{v}$ to see it back as an ω-sequence $\boldsymbol{V} = \boldsymbol{s}_{n+1}\boldsymbol{s}_{n+2}\cdots$ over Y. Since $\boldsymbol{v}$ is indecomposable, $\boldsymbol{s}_{n+1}$ embeds in some finite prefix of what remains of $\boldsymbol{V}$, *i.e.* $\boldsymbol{s}_{n+1} \leq \boldsymbol{s}_{n+2}\cdots\boldsymbol{s}_m$ for some $m < \omega$. But since $\boldsymbol{s}_{n+1}$ is an infinite sequence, by the pigeon-hole principle an infinite suffix of $\boldsymbol{s}_{n+1}$ embeds in $\boldsymbol{s}_i$ for some $n+2 \leq i \leq m$, and since $\boldsymbol{s}_{n+2}$ is indecomposable, $\boldsymbol{s}_{n+1} \leq \boldsymbol{s}_i$. This is an increasing pair of $\boldsymbol{S}$, which proves Y is a WQO.

$3 \Rightarrow 10$: This implication follows from an analysis of the embedding relation between ordinal sequences which is very similar to the one we conducted in Proposition 9.1.1, but we failed to provide a proof that uses this analysis without replaying it. As previously observed, the "unflattening" defines a reflection from X^{ω^2} to $(X^\omega)^\omega$ which is in general not an embedding. Therefore, the image of the embedding quasi-ordering on X^{ω^2} is an extension of the embedding on $(X^\omega)^\omega$. However, if extensions of a WQO are WQOs, extensions of a well-founded quasi-orderings may not be well-founded. As a result, we cannot deduce the well-foundedness of X^{ω^2} from the well-foundedness of $(X^\omega)^\omega$ (which follows from Section 9.1 when X^ω is a WQO).

Given a sequence $\boldsymbol{s} \in X^{\omega^2}$, we denote by $\boldsymbol{s}_i \in X^\omega$ the ω-sequences such that $\boldsymbol{s} = \boldsymbol{s}_1\boldsymbol{s}_2\cdots$, that is $\forall i, j \in \mathbb{N}$, $\boldsymbol{s}_i(j) = \boldsymbol{s}(\omega \cdot i + j)$. Recall from Section 9.1 the notion of *tail* of an ω-sequence. The *tail* of a sequence $\boldsymbol{s} \in X^{\omega^2}$, denoted $\mathcal{D}(\boldsymbol{s}) \in Down(Down(X))$, is defined by $\mathcal{D}(\boldsymbol{s}) = \bigcap_{i<\omega} \bigcup_{j\geq i} {\downarrow} D(\boldsymbol{s}_i)$, where ${\downarrow} D$ denotes $\{E \in Down(X) \mid E \subseteq D\}$, that is the downward-closure is taken of $Down(Down(X))$, and ${\downarrow} D$ is a principal ideal of downward-closed sets. Assuming $Down(X)$ is a WQO, $Down(Down(X))$ is well-founded ((WQO7)) and a infinite increasing subsequence can be extracted from the ω-sequence $(D(\boldsymbol{s}_i))_{i<\omega}$, implying that $\mathcal{D}(\boldsymbol{s}) \neq \emptyset$. It remains to show that whenever two ω^2-sequences $\boldsymbol{s}$ and $\boldsymbol{t}$ are ordered $\boldsymbol{s} \leq \boldsymbol{t}$, then $\mathcal{D}(\boldsymbol{s}) \subseteq \mathcal{D}(\boldsymbol{t})$. Let $\boldsymbol{s}, \boldsymbol{t}$ be such two sequences and $f : \boldsymbol{\omega^2} \to \boldsymbol{\omega^2}$ be a witness of the embedding $\boldsymbol{s} \leq \boldsymbol{t}$. Let $i \in \mathbb{N}$. The image of $f(\omega \cdot (i+1))$ is in $\boldsymbol{\omega^2}$, thus it is some $\omega \cdot n + m$. This means that $\boldsymbol{s}_i \leq \boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_n\boldsymbol{t}_{n+1}$. Therefore, by the pigeon-hole principle, there exists $j \leq n+1$ and an infinite suffix $\boldsymbol{s}'_i$ of $\boldsymbol{s}_i$ such that $\boldsymbol{s}'_i \leq \boldsymbol{t}_j$. Observe that $D(\boldsymbol{s}'_i) = D(\boldsymbol{s}_i)$, and by Proposition 9.1.1, $D(\boldsymbol{s}'_i) \subseteq D(\boldsymbol{t}_j)$.

In the end, if $D \in \mathcal{D}(\boldsymbol{s})$, it is covered by infinitely many $D(\boldsymbol{s}_i)$, and thus by infinitely many $D(\boldsymbol{t}_j)$, and is thus a member of $\mathcal{D}(\boldsymbol{t})$.

Lastly, for Item 11, we cannot provide a proof since we did not define Rado's structure. $11 \iff 8$: is proved in the Rado's original article [38]. The right-to-left direction can also be found in [43, 11] (with a nice illustration of Rado's structure in [43]). Otherwise, a direct proof of $11 \iff 6$: can be found in [44]. □

Many of the equivalences above generalize to countable ordinals.

Proposition 9.3.4. *Given an indecomposable countable ordinal α, $(X, \leq)$ is α-WQO if and only if $(X^{<\alpha}, \leq)$ is WQO [11, 45].*

The above generalizes items 1 and 9. Items 2, 3,4, 5, 6 and 7 are generalized as the equivalence between [46]:

- $(X, \leq)$ *is* $(\alpha \cdot \omega)$*-WQO,*
- $(\mathit{Idl}(X), \subseteq)$ *is* α*-WQO,*
- $(\mathit{Down}(X), \subseteq)$ *is* α*-WQO,*
- $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$ *is* α*-WQO,*
- $(\mathcal{P}(X), \sqsubseteq_{\mathcal{S}})$ *is* α*-WQO,*
- $(\mathcal{P}_f(X), \sqsubseteq_{\mathcal{S}})$ *is* α*-WQO,*
- $(\mathit{Up}(X), \supseteq)$ *is* α*-WQO.*

The generalization of Item 11 is what led Nash-Williams to the original definition of BQOs, with the notion of blocks. This intuition of blocks is formalized in [42], Theorem III-3.3. Concerning Item 10, the implication X^α well-founded $\Rightarrow X$ α-WQO is proved in [11], the general proof being as simple as the one above. However, we were unable to find mention of the converse implication in the literature, and have been unable to prove it or disprove it. It seems possible to generalize the proof given above for $\alpha = \omega^2$ to indecomposable ordinals of the form $\omega^{\alpha+1}$. Indeed, we essentially used that if X is $\omega^{\alpha+1}$-WQO, then $Y \stackrel{\text{def}}{=} X^{\omega^\alpha}$ is WQO, and then $X^{\omega^{\alpha+1}}$ can be approximated by Y^ω. But in the case X is ω^λ-WQO, with λ a limit ordinal, we cannot use the same methods.

Finally, let us mention that all the order-construction mentioned in the previous chapters that preserve WQO also preserve α-WQO. For $\alpha = \omega^2$, this will be a trivial consequence of our analysis in the next section. In the general case, and for more details on BQO theory, we redirect the reader to the following surveys [43, 45, 42, 11]

9.4 Ideally Effective ω^2-WQOs

In this section, we extend our notion of effectiveness to ω^2-WQOs.

If $(X, \leq)$ is an ordinal, then so is $(\mathit{Idl}(X), \subseteq)$ (Section 3.2.3), and therefore it is ideally effective (successor of a recursive ordinal is recursive). Similarly, if $(X, \leq)$ is finite, then $(\mathit{Idl}(X), \subseteq)$ is isomorphic to $(X, \leq)$ and is thus ideally effective. Recall that $(\mathit{Down}(X), \subseteq)$ is ideally effective whenever $(\mathit{Idl}(X), \subseteq)$ is (Section 7.3). Thus in these cases, $(X^\omega, \leq_\omega)$ is ideally effective. It remains to show that the constructions we have considered will preserve these properties. The properties we are formally interested in are formalized in the next definition.

Definition 9.4.1. *A WQO $(X, \leq)$ is Idl^2-*effective *if:*

- $(X, \leq)$ *is a* ω^2*-WQO, that is* $(Idl(X), \subseteq)$ *is a WQO,*
- $(X, \leq)$ *is an ideally effective WQO,*
- $(Idl(X), \subseteq)$ *is an ideally effective WQO when representing elements of* $Idl(X)$ *(axiom* (XR)*) using the same representation used for ideals of* X *(axiom* (IR)*).*
- *The* flattening *function defined below is computable:*

$$\begin{cases} Idl(Idl(X)) & \rightarrow & Idl(X) \\ \boldsymbol{J} & \mapsto \bigcup & \boldsymbol{J} \end{cases}$$

where the output is represented according to axiom (IR) *for* X.

We will also say that $(X, \leq)$ *is an ideally effective* ω^2*-WQO.*

The basic WQOs from Section 3.2 are examples of Idl^2-effective WQOs.

- For a finite WQO $(X, \leq)$, ideals of X are isomorphic to X itself, and thus so are ideals of ideals. We thus represent the three sets with the same representation, computability of set-theoretic operations then follow from the ideal effectiveness of finite WQOs. Finally, the flattening function is the identity.
- The set of ideals of an ordinal $(\boldsymbol{\alpha}, \leq)$ ordered with inclusion is isomorphic to $\boldsymbol{\alpha} + \boldsymbol{1}$. Therefore, the set of ideals of ideals of $\boldsymbol{\alpha}$ is isomorphic to $\boldsymbol{\alpha} + \boldsymbol{2}$, and every operation is computable. For the flattening function, given a successor ordinal $\beta + 1 \in \boldsymbol{\alpha} + \boldsymbol{2}$, $\bigcup(\boldsymbol{\beta} + \boldsymbol{1}) = \boldsymbol{\beta}$, and given a limit ordinal $\lambda \in \boldsymbol{\alpha} + \boldsymbol{2}$, $\bigcup \boldsymbol{\lambda} = \boldsymbol{\lambda}$.

Next, we prove that when $(X, \leq)$ is an Idl^2-effective WQO, $(X^\omega, \leq_\omega)$ is ideally effective. The previous definition has been designed to this end.

Proposition 9.4.2. *Let* $(X, \leq)$ *be an* Idl^2*-effective WQO. Then* $(Down(X), \subseteq)$ *(and therefore* $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$*) and* $(X^\omega, \leq_\omega)$ *are ideally effective.*

Proof. The ideal effectiveness of $(Down(X), \subseteq)$ follows from the isomorphism between the latter and $(\mathcal{P}_f(Idl(X)), \sqsubseteq_{\mathcal{H}})$, as a result of Section 7.3.

To establish the ideal effectiveness of $(X^\omega, \leq_\omega)$, we show that the requirements identified in Section 9.1 are met. Obviously, $(X, \leq)$ is ideally effective, and $(Down(X), \subseteq)$ as well according to what preceeds. Since we have shown that $(Down(X), \subseteq)$ is ideally effective using the standard representation for downward-closed sets (*i.e.* finite sets of ideals), the *transfer* function is trivial. Finally, we have to show that the following function is computable:

$$\begin{cases} Idl(Down(X)) & \rightarrow & Down(X) \\ \boldsymbol{J} & \mapsto & \bigcup \boldsymbol{J} \end{cases}$$

However, Definition 9.4.1 only provides the computability of this function:

$$\begin{cases} Idl(Idl(X)) & \rightarrow & Idl(X) \\ \boldsymbol{J} & \mapsto & \bigcup \boldsymbol{J} \end{cases}$$

Recall from Section 7.3 that ideals of $Down(X) \equiv \mathcal{P}_f(Idl(X))$ are of the form $\mathcal{P}_f(\boldsymbol{D})$ for $\boldsymbol{D} \in Down(Idl(X))$, and therefore simply encoded as elements of $Down(Idl(X))$. Let $\boldsymbol{J}$ be an actual ideal of $Down(X)$ (*i.e.* its semantic), and $\boldsymbol{D}$ be its representation (its syntax). Then, $\bigcup \boldsymbol{J} = \bigcup \boldsymbol{D}$. Indeed, $\boldsymbol{D}$ stands for $\mathcal{P}_f(\boldsymbol{D}) = \{E \in \mathcal{P}_f(Idl(X)) \mid E \subseteq D\}$, and an element $E \in \mathcal{P}_f(Idl(X))$ actually stands for $\bigcup E$, that is the downward-closed set obtained as the union of the ideals members of E. Therefore, $\boldsymbol{J} = \{\bigcup E \mid E \in \mathcal{P}_f(Idl(X)) \text{ and } E \subseteq D\}$.

Let $x \in \bigcup \boldsymbol{J}$, there exists a downward-closed set $F \in Down(X)$ such that $x \in F \in \boldsymbol{J}$. Therefore, x is in some ideal $I \in Idl(X)$ which appears in the canonical decomposition of F, and by definition, $I \in \boldsymbol{D}$. Thus $x \in I \in \boldsymbol{D}$, proving $\bigcup \boldsymbol{J} \subseteq \bigcup \boldsymbol{D}$.

For the other direction, let $x \in \boldsymbol{D}$, there exists $I \in Idl(X)$ such that $x \in I \in \boldsymbol{D}$. Therefore $\{I\} \subseteq \boldsymbol{D}$ and the downward-closed set I is a member of $\boldsymbol{J}$. It follows that $x \in I \in \boldsymbol{J}$ and $\bigcup \boldsymbol{D} \subseteq \bigcup \boldsymbol{J}$.

It now remains to prove that $\bigcup \boldsymbol{D}$ is computable. Remember $\boldsymbol{D} \in Down(Idl(X))$, and it therefore has an ideal decomposition $\boldsymbol{D} = \boldsymbol{I}_1 \cup \cdots \boldsymbol{I}_n$ for some $\boldsymbol{I}_1, \ldots, \boldsymbol{I}_n \in Idl(Idl(X))$. We can similarly prove $\bigcup \boldsymbol{D} = \bigcup_{i=1}^n \bigcup \boldsymbol{I}_i$, and given $\boldsymbol{I} \in Idl(Idl(X))$, $\bigcup \boldsymbol{I}$ is computable by Definition 9.4.1. □

We now want to show that our algebra of ideally effective WQOs is actually an algebra of Idl^2-effective WQOs, which means that $(X^\omega, \leq_\omega)$ is ideally effective for every WQO $(X, \leq)$ in this algebra. For this, it suffices to show that our constructions preserve not only ideal effectiveness, but Idl^2-effectiveness.

Definition 9.4.3. *An order-theoretic construction C is Idl^2-*effective *if, for every Idl^2-effective WQOs $(X_1, \leq_1), \ldots, (X_n, \leq_n)$:*

- *$C((X_1, \leq_1), \ldots, (X_n, \leq_n))$ is an Idl^2-effective WQO, and*
- *Full presentations for $C((X_1, \leq_1), \ldots, (X_n, \leq_n))$ and $Idl(C((X_1, \leq_1), \ldots, (X_n, \leq_n)))$ are computable from full presentations of the $(X_i, \leq_i)$ and the $(Idl(X_i), \subseteq)$.*
- *The flattening function for $C((X_1, \leq_1), \ldots, (X_n, \leq_n))$ is computable from flattening functions for the $(X_i, \leq_i)$.*

Theorem 9.4.4. *The following constructions are Idl^2-effective:*

- *Disjoint Sum (Section 5.1),*
- *Lexicographic Sum (Section 5.2),*
- *Cartesian Product with Dickson's quasi-ordering (Section 5.3),*
- *Finite Sequences with Higman's quasi-rdering (Section 6.1),*
- *Finite Sequences with Stuttering quasi-ordering (Section 6.2),*
- *Finite Multisets with Embedding quasi-rdering (Section 7.1)*
- *Finite Powerset with Hoare quasi-rdering (Section 7.3),*

Proof. Given Idl^2-effective WQOs $(X_1, \leq_1), \ldots, (X_n, \leq_n)$, we want to prove that $C((X_1, \leq_1), \ldots, (X_n, \leq_n))$ is an Idl^2-effective WQO. The second requirement of Definition 9.4.1 that $C((X_1, \leq_1), \ldots, (X_n, \leq_n))$ is an ideally effective WQO, has already been proved in the chapter dedicated to construction C. Therefore we focus on proving the third requirement: $\mathit{Idl}(C((X_1, \leq_1), \ldots, (X_n, \leq_n)))$ is an ideally effective WQO, showing in particular that it is a WQO, fulfilling the first requirement of Definition 9.4.1. The second requirement of Definition 9.4.3 on computability of full presentations will always result from prior analysis and will not be mentioned. Finally, we will also argue that the flattening function is computable in each case.

Let $(X, \leq)$, $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be ideally effective ω^2-WQOs. In particular, $(\mathit{Idl}(X), \subseteq)$ and $(\mathit{Down}(X), \subseteq)$ are ideally effective.

Disjoint Sum: Since $\mathit{Idl}(X_1)$ and $\mathit{Idl}(X_2)$ are ideally-effective, Section 5.1 proves that $(\mathit{Idl}(X_1 \sqcup X_2), \subseteq) \equiv (\mathit{Idl}(X_1), \subseteq) \sqcup (\mathit{Idl}(X_2), \subseteq)$ is ideally effective. The flattening function for the sum simply is the sum of the flattening functions for X_1 and X_2: given an ideal of $\mathit{Idl}(\mathit{Idl}(X_1 \sqcup X_2))$, it is of the form $\langle i, \boldsymbol{I} \rangle$ for some $i \in \{1, 2\}$ and some $\boldsymbol{I} \in \mathit{Idl}(\mathit{Idl}(X_i))$. Therefore, $\bigcup \langle i, \boldsymbol{I} \rangle = \langle i, \bigcup \boldsymbol{I} \rangle$.

Lexicographic Sum: Here also, the ideal effectiveness of $\mathit{Idl}(X_1 \oplus X_2) \equiv \mathit{Idl}(X_1 \subseteq) \oplus \mathit{Idl}(X_2, \subseteq)$ follows from Section 5.2. The flattening function is exactly the same as before, but keep in mind that the representation $\langle 2, \bigcup \boldsymbol{I} \rangle$ actually stands for the set $X_1 \cup \bigcup \boldsymbol{I}$.

Cartesian Product with Dickson's Quasi-Ordering: Again, the ideal effectiveness of $\mathit{Idl}(X_1 \times X_2) \equiv \mathit{Idl}(X_1, \subseteq) \times \mathit{Idl}(X_2, \subseteq)$ follows from Section 5.3. Given $\boldsymbol{I} \in \mathit{Idl}(\mathit{Idl}(X_1))$ and $\boldsymbol{J} \in \mathit{Idl}(\mathit{Idl}(X_2))$, $\bigcup(\boldsymbol{I} \times \boldsymbol{J}) = (\bigcup \boldsymbol{I}) \times (\bigcup \boldsymbol{J})$.

Finite Sequences with Higman's Quasi-Ordering: In this case, the set $\mathit{Idl}(X^*)$ is in bijection with $(\mathit{Idl}(X) \sqcup \mathit{Down}(X))^*$, but inclusion on the former does not correspond to the natural quasi-ordering $\leq_*$ $(\leq_\sqcup)$ on the latter, instead it corresponds to an extension of this natural QO.

The Atom Construction Preserves Ideal Effectiveness: We first deal with the atoms $\mathit{Atm}(X)$ (see Section 6.1 for the definition): the set of atoms is isomorphic to an extension of $(\mathit{Idl}(X) \sqcup \mathit{Down}(X), \subseteq_\sqcup)$. Actually, the atom $\emptyset^*$ can always be removed from an atom decomposition of an ideal of $\mathit{Idl}(X^*)$ (see Definition 6.1.3). We therefore work with $(\mathit{Idl}(X) \sqcup \mathit{Down}'(X), \subseteq_\sqcup)$ instead, where $\mathit{Down}'(X) \stackrel{\text{def}}{=} \mathit{Down}(X) \smallsetminus \{\emptyset\}$. Let $\sqsubseteq$ be the image of the inclusion quasi-ordering on the atoms of X^* by the isomorphism above, that is:

$$
\begin{aligned}
\langle 1, I \rangle \sqsubseteq \langle 1, J \rangle &\iff I + \epsilon \subseteq J + \epsilon && \iff I \subseteq J \\
\langle 1, I \rangle \sqsubseteq \langle 2, D \rangle &\iff I + \epsilon \subseteq D^* && \iff I \subseteq D \\
\langle 2, D \rangle \sqsubseteq \langle 2, D' \rangle &\iff D^* \subseteq D'^* && \iff D \subseteq D' \\
\langle 2, D \rangle \sqsubseteq \langle 1, I \rangle &\iff D^* \subseteq I + \epsilon && \text{which is never true } (D \neq \emptyset
\end{aligned}
$$

With this notation, $(Atm(X), \subseteq)$ is isomorphic to $(Idl(X) \sqcup Down(X), \sqsubseteq)$. We show that the latter is ideally effective using Section 4.1, since $\subseteq_\sqcup \subseteq \sqsubseteq$: it suffices to show that function $\mathcal{C}l_\mathrm{F}$ and $\mathcal{C}l_\mathrm{I}$ are computable. To compute $\mathcal{C}l_\mathrm{F}$, use the following equations:

$$\mathcal{C}l_\mathrm{F}(\langle 1, I\rangle) = \uparrow_\sqcup\langle 1, I\rangle \cup \uparrow_\sqcup\langle 2, I\rangle$$
$$\mathcal{C}l_\mathrm{F}(\langle 2, D\rangle) = \uparrow_\sqcup\langle 2, D\rangle$$

Regarding function $\mathcal{C}l_\mathrm{I}$, recall from Section 5.1 that the set of ideals of $(Idl(X) \sqcup Down(X), \leq_\sqcup)$ is the disjoint sum of the set of ideals of $Idl(X)$ and of the set of ideals of $Down(X)$, that is, an ideal of $(Idl(X) \sqcup Down(X), \leq_\sqcup)$ is either of the form $\langle 1, \boldsymbol{I}\rangle$ for $\boldsymbol{I} \in Idl(Idl(X))$ or of the form $\langle 2, \boldsymbol{J}\rangle$ for $\boldsymbol{J} \in Idl(Down(X))$. Note that $Down(X)$ is ideally effective according to Proposition 9.4.2.

We can now show that $\mathcal{C}l_\mathrm{I}$ is computable:

$$\mathcal{C}l_\mathrm{I}(\langle 1, \boldsymbol{I}\rangle) \stackrel{\text{def}}{=} \downarrow_\sqsubseteq\langle 1, \boldsymbol{I}\rangle = \langle 1, \boldsymbol{I}\rangle$$
$$\mathcal{C}l_\mathrm{I}(\langle 2, \boldsymbol{J}\rangle) \stackrel{\text{def}}{=} \downarrow_\sqsubseteq\langle 2, \boldsymbol{J}\rangle = \langle 2, \boldsymbol{J}\rangle \cup \bigcup_{i=1}^{n}\langle 1, \boldsymbol{I}_i\rangle$$

where $\bigcup_{i=1}^n \boldsymbol{I}_i$ is the ideal decomposition of $\bigcup \boldsymbol{J}$ where $\boldsymbol{J}$ is seen as an ideal of $\mathcal{P}_f(Idl(X))$.

Let us argue the correctness and computability of the second equation. The ideal $\boldsymbol{J} \in Idl(Down(X))$ is actually encoded as an element of $Idl(\mathcal{P}_f(Idl(X)))$, according to our representation of downward-closed sets of X. Therefore, $\bigcup \boldsymbol{J} = \bigcup_{S \in \boldsymbol{J}} S$ is a downward-closed set of ideals of X, which admits a decomposition into ideals of ideals of X. This is precisely the decomposition $\bigcup_{i=1}^n \boldsymbol{I}_i$ we use in the equation above.

It remains to show that this particular decomposition can be computed from the actual encoding of $\boldsymbol{J}$. Remember from Section 7.3 that the ideals of $\mathcal{P}_f(Idl(X))$ are of the form $\mathcal{P}_f(\boldsymbol{D})$ for $\boldsymbol{D} \in Down(Idl(X))$, and therefore $\boldsymbol{J}$ is actually encoded as $\boldsymbol{D}$ (as in the proof of Proposition 9.4.2). We now prove that semantically, $\boldsymbol{D} = \bigcup_{S \in \boldsymbol{J}} S$, which concludes the proof since the ideal decomposition of $\boldsymbol{D}$ can be computed.

Let $I \in \boldsymbol{D}$, then $S \stackrel{\text{def}}{=} \{I\} \in \mathcal{P}_f(\boldsymbol{D}) = \boldsymbol{J}$ which proves that $I \in S \in \boldsymbol{J}$. Conversely, given some $I \in \boldsymbol{J}$, there exists some $S \in \boldsymbol{J} = \mathcal{P}_f(\boldsymbol{D})$ such that $I \in S \subseteq \boldsymbol{D}$. Therefore, $I \in \boldsymbol{D}$.

In conclusion, functions $\mathcal{C}l_\mathrm{I}$ and $\mathcal{C}l_\mathrm{F}$ being computable, $(Atm(X), \sqsubseteq)$ is ideally effective.

Back to the Idl^2-effectiveness of X^*, the ideals of X^* are finite sequences of atoms. Unfortunately, the quasi-ordering on these sequences does not coincide with one of the three quasi-ordering on X^* that we have shown to be ideally effective. Indeed, when embedding a sequence of atom into another, we are allowed to stutter (in the sense of Section 6.2) on atoms of the form $\langle 2, D\rangle$ for $D \in Down(X)$, but not on atoms of the form $\langle 1, I\rangle$ for $I \in Idl(X)$. Hence, we introduce a quasi-ordering on sequences which generalizes both the Higman quasi-ordering and the stuttering quasi-ordering. We call it the *partial stuttering quasi-ordering*, and we now show that it is an ideally effective construction (for any ideally effective WQO $(X, \leq)$, not only for atoms, which is the case we are ultimately interested in).

The Partial Stuttering Quasi-Ordering is Ideally Effective: Given an ideally effective WQO $(X, \leq)$ and an upward-closed set $A \subseteq X$, define the *partial stuttering quasi-ordering on* A by:

$$x_1 \cdots x_n \leq_{\mathrm{st}}^A y_1 \cdots y_m \overset{\text{def}}{\Leftrightarrow} \exists f : [n] \to [m].\ \forall i \in [n].\ x_i \leq y_{f(i)} \text{ and } \forall i \neq j \in [n].\ f(i) = f(j) \Rightarrow y_{f(i)} \in A.$$

This QO generalizes both the Higman quasi-ordering (from Section 6.1) and the stuttering quasi-ordering (from Section 6.2): if $A = \emptyset$, then $\leq_{\mathrm{st}}^A = \leq_*$ and when $A = X$, $\leq_{\mathrm{st}}^A = \leq_{\mathrm{st}}$. In all generality, the following holds: $\leq_* \subseteq \leq_{\mathrm{st}}^A \subseteq \leq_{\mathrm{st}}$. In particular, this QO is an extension of $\leq_*$, and we can use Section 4.1 to prove it ideally effective.

Since A is upward-closed, $\leq_{\mathrm{st}}^A$ is transitive: it suffices to compose the witnesses of the embeddings. Moreover, it is also still possible to concatenate witnesses, that is if $u \leq_{\mathrm{st}}^A u'$ and $v \leq_{\mathrm{st}}^A v'$ then $u \cdot v \leq_{\mathrm{st}}^A u' \cdot v'$ ($\leq_{\mathrm{st}}^A$ is *compatible* with concatenation). In particular, it is still the case that $(\downarrow_{\mathrm{st}}^A \boldsymbol{P}_1) \cdot (\downarrow_{\mathrm{st}}^A \boldsymbol{P}_2) = \downarrow_{\mathrm{st}}^A (\boldsymbol{P}_1 \cdot \boldsymbol{P}_2)$, for $\boldsymbol{P}_1, \boldsymbol{P}_2$ ideals of $(X^*, \leq_*)$. Therefore, it only remains to compute $\mathcal{C}l_{\mathrm{I}}$ on atoms.

$$\begin{aligned} \mathcal{C}l_{\mathrm{I}}(I + \epsilon) &= I^* && \text{when } I \cap A \neq \emptyset \\ \mathcal{C}l_{\mathrm{I}}(I + \epsilon) &= I + \epsilon && \text{otherwise} \\ \mathcal{C}l_{\mathrm{I}}(D^*) &= D^* \end{aligned}$$

Proof. If $A \cap I \neq \emptyset$ then:

- $\downarrow_{\mathrm{st}}^A I \subseteq I^*$: if $\boldsymbol{u} \leq_{\mathrm{st}}^A \boldsymbol{x}$ for some $x \in I$, then $\boldsymbol{u} \in (\downarrow x)^* \subseteq I^*$. The case $\boldsymbol{u} \leq_{\mathrm{st}}^A \epsilon$ is trivial.
- $I^* \subseteq \downarrow_{\mathrm{st}}^A I$: given $\boldsymbol{u} \in I^*$, since $\boldsymbol{u}$ is finite and I is directed, we can find $x \in I$ such that x is greater than every element of $\boldsymbol{u}$. Moreover, since I is directed, there exists $z \in I$ which is greater than all elements of $\boldsymbol{u}$ (greater than x) and which is in A (pick any $y \in A \cap I$). It satisfies $\boldsymbol{u} \leq_{\mathrm{st}}^A \boldsymbol{z}$.

If $A \cap I = \emptyset$ then: for any $\boldsymbol{u} \leq_{\mathrm{st}}^A \boldsymbol{x}$ for some $x \in I$, since $x \notin A$, $|\boldsymbol{u}| \leq 1$ and therefore $\boldsymbol{u} = \epsilon$, or $\boldsymbol{u} = \boldsymbol{y}$ for $y \leq x \in I$. The other inclusion is trivial. So is the third equation. ☐

Observe that the condition $I \cap U \neq \emptyset$ for $I \in \mathit{Idl}(X)$ and $U \in \mathit{Up}(X)$ is decidable. Indeed, $I \cap \uparrow x \neq \emptyset \iff x \in I$: the right-to-left direction is trivial, and if there exists $y \in I \cap \uparrow x$, then $x \leq y \in I$ which implies $x \in I$ since I is downward-closed. Emptiness of the intersection with an upward-closed is then tested by distributing the intersection over the unions. Therefore, $\mathcal{C}l_{\mathrm{I}}$ is computable.

As in Section 6.2, function $\mathcal{C}l_{\mathrm{F}}$ is more complicated: let $\boldsymbol{u} = x_1 \cdots x_n$,

$$\mathcal{C}l_{\mathrm{F}}(\boldsymbol{u}) = \uparrow_{\mathrm{st}} \boldsymbol{u} = \uparrow_* \left\{ y_1 \cdots y_k \;\middle|\; \begin{array}{c} 0 \leq k \leq n \\ 0 = i_0 < i_1 < \cdots < i_k = n \\ \forall j \in [k].\ y_i \in \min(\bigcap_{i_{j-1} < \ell \leq i_j} \uparrow_X x_\ell) \\ \forall j \in [k].\ (i_j = i_{j-1} + 1 \vee y_j \in A) \end{array} \right\}$$

Remember the intuition given in Section 6.2: if $\boldsymbol{u} \leq_{\mathrm{st}} bw$, we can extract the image $\boldsymbol{v} = y_1 \cdots y_k$ of an embedding witnessing $\boldsymbol{u} \leq_{\mathrm{st}} \boldsymbol{w}$ ($\boldsymbol{v}$ is a subsequence of $\boldsymbol{w}$). This induces a factorization $\boldsymbol{u} = \boldsymbol{u}_1 \cdots \boldsymbol{u}_k$ of $\boldsymbol{u}$ where $\boldsymbol{u}_i \leq_{\mathrm{st}} y_i$ for all $i \in [k]$. Now for $\leq_{\mathrm{st}}^A$, we must also ensures that either $y_i \in A$ (it can embeds several elements), or $|\boldsymbol{u}_i| = 1$. The proof is left to the reader.

In conclusion, $\mathcal{C}l_{\mathrm{F}}$ is computable as well, which proves that $(X^*, \leq_{\mathrm{st}}^A)$ is ideally effective for any $A \in \mathit{Up}(X)$ whenever $(X, \leq)$ is ideally effective. In our case, $(\mathit{Idl}(X^*), \subseteq)$ is isomorphic to $(\mathit{Atm}(X)^*, \leq_{\mathrm{st}}^{\langle 2, \mathit{Down}'(X)\rangle})$, where as expected, $\langle 2, \mathit{Down}'(X)\rangle$ designates $\{\langle 2, D\rangle \mid D \in \mathit{Down}'(X)\} \in \mathit{Up}(\mathit{Atm}(X))$. The latter is ideally-effective according to what preceeds.

To establish the Idl^2-effectiveness of $(X^*, \leq_*)$, it remains to prove the computability of the flattening function. Let $\boldsymbol{\mathcal{P}} \in \mathit{Idl}(\mathit{Idl}(X^*))$. It is encoded as an ideal of $(\mathit{Atm}(X)^*, \leq_{\mathrm{st}}^{\langle 2, \mathit{Down}'(X)\rangle})$. These ideals are themselves encoded as sequences of some particular atoms of $\mathit{Atm}(X)$, which we will subsequently call *higher atoms*. Given $\boldsymbol{I} \in \mathit{Idl}(\mathit{Idl}(X))$, then $\langle 1, \boldsymbol{I}\rangle \in \mathit{Idl}(\mathit{Atm}(X))$ and $\langle 1, \boldsymbol{I}\rangle + \epsilon$ is a higher atom. If $\boldsymbol{D} \in \mathit{Down}(\mathit{Idl}(X))$, then $\langle 1, \boldsymbol{D}\rangle \in \mathit{Down}(\mathit{Atm}(X))$ and $\langle 1, \boldsymbol{D}\rangle^*$ is a higher atom. Lastly, given $\boldsymbol{E} \in \mathit{Down}(\mathit{Down}(X))$, $\langle 2, \boldsymbol{E}\rangle \in \mathit{Down}(\mathit{Atm}(X))$ and $\langle 2, \boldsymbol{E}\rangle^*$ is a higher atom. Moreover, these are the only types of higher atoms.

Now, any $\boldsymbol{\mathcal{P}} \in \mathit{Idl}(\mathit{Idl}(X^*))$ is encoded as a product $\boldsymbol{\mathcal{A}}_1 \cdots \boldsymbol{\mathcal{A}}_n$ of higher atoms, and the flattening of $\boldsymbol{\mathcal{P}}$ is the product of flattening of its higher atoms:

$$\bigcup \boldsymbol{\mathcal{P}} = (\bigcup \boldsymbol{\mathcal{A}}_1) \cdots (\bigcup \boldsymbol{\mathcal{A}}_n)$$

It thus remain to prove that the flattening of higher atoms is computable. Let $\boldsymbol{\mathcal{A}}$ be an higher atom.

- If $\boldsymbol{\mathcal{A}} = \langle 1, \boldsymbol{I}\rangle + \epsilon$, then $\bigcup \boldsymbol{\mathcal{A}} = \langle 1, \bigcup \boldsymbol{I}\rangle$.
- If $\boldsymbol{\mathcal{A}} = \langle 1, \boldsymbol{D}\rangle^*$, then $\bigcup \boldsymbol{\mathcal{A}} = \langle 1, \bigcup \boldsymbol{D}\rangle^*$. Indeed, if $\boldsymbol{u} \in \bigcup \boldsymbol{\mathcal{A}}$ then there exists some ideals $I_1, \ldots, I_n \in \boldsymbol{D} \subseteq \mathit{Idl}(X)$ for $n = |u|$ such that $\boldsymbol{u} \in (I_1 + \epsilon) \cdots (I_n + \epsilon)$. But since for every i, $I_i \subseteq \bigcup \boldsymbol{D}$, $\boldsymbol{u} \in (\bigcup \boldsymbol{D})^*$.

  Conversely, let $\boldsymbol{u} = x_1 \cdots x_n \in (\bigcup \boldsymbol{D})^*$. In particular, for each i, $x_i \in \bigcup \boldsymbol{D}$, and there exists an ideal such that $x_i \in I_i \in \boldsymbol{D}$. Therefore, $\boldsymbol{u} \in (I_1 + \epsilon) \cdots (I_n + \epsilon) \in \langle 1, \boldsymbol{D}\rangle^*$.

  Besides, we can compute $\boldsymbol{D} = \boldsymbol{I}_1 \cup \cdots \cup \boldsymbol{I}_m$ the ideal decomposition of $\boldsymbol{D}$, with $\boldsymbol{I}_i \in \mathit{Idl}(\mathit{Idl}(X))$, and we have

  $$\bigcup \boldsymbol{D} = (\bigcup \boldsymbol{I}_1) \cup \cdots \cup (\bigcup \boldsymbol{I}_m)$$

  This last expression is computable using the flattening function for $\mathit{Idl}(\mathit{Idl}(X))$.
- Lastly, if $\boldsymbol{\mathcal{A}} = \langle 2, \boldsymbol{E}\rangle^*$ for some $\boldsymbol{E} \in \mathit{Down}(\mathit{Down}(X))$, then $\bigcup \boldsymbol{\mathcal{A}} = \langle 2, \bigcup \boldsymbol{E}\rangle^*$. The proof is similar to the previous one. Besides, the expression $\bigcup \boldsymbol{E}$ for $\boldsymbol{E} \in \mathit{Down}(\mathit{Down}(X))$ has already been shown computable in Proposition 9.4.2.

Finite Sequences with Stuttering: In this case, $\mathit{Idl}(X^*, \leq_{\mathrm{st}})$ is isomorphic to $(\mathit{Down}(X)^*, \subseteq_{\mathrm{st}})$. which is ideally effective since $(\mathit{Down}(X), \subseteq)$ is (*cf.* Section 6.2).

Note that the quasi-ordering $\leq_{\mathrm{st}}^A$ on X^* introduced in the previous paragraph generalizes both $\leq_*$ ($A = \emptyset$) and $\leq_{\mathrm{st}}$ ($A = X$). Therefore, the Idl^2-effectiveness of these two constructions follows from the Idl^2-effectiveness of $(X^*, \leq_{\mathrm{st}}^A)$, which can be proved following the lines of the previous case.

Finite Multisets with the Embedding Quasi-Ordering: In this case, $\mathit{Idl}(X^{\circledast}, \leq_{\mathrm{emb}})$ is isomorphic to an extension of $(\mathit{Idl}(X)^{\circledast} \times \mathit{Down}(X), \leq_\times)$. The latter is ideally effective according to Sections 5.3 and 7.1. We show that its extension is ideally effective using Section 4.1.

Let $\sqsubseteq$ denotes the image of $\subseteq$ by the function that maps an ideal $\boldsymbol{I} \in \mathit{Idl}(X^{\circledast})$ to $\langle \boldsymbol{B}, D\rangle \in \mathit{Idl}(X)^{\circledast} \times \mathit{Down}(X)$ where $\boldsymbol{I} = \downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$, that is:

$$\langle \boldsymbol{B}, D\rangle \sqsubseteq \langle \boldsymbol{C}, E\rangle \overset{\text{def}}{\Leftrightarrow} \boldsymbol{B} \setminus E \subseteq_{\mathrm{emb}} \boldsymbol{C} \wedge D \subseteq E$$

Function $\mathcal{C}l_{\mathrm{F}}$ is easily seen computable: given $\langle \boldsymbol{B}, D\rangle$ an ideal of $(\mathit{Idl}(X)^{\circledast} \times \mathit{Down}(X), \leq_\times)$,

$$\mathcal{C}l_{\mathrm{F}}(\langle \boldsymbol{B}, D\rangle) = \bigcup_{\boldsymbol{C} \subset \boldsymbol{B}} \uparrow_\times \langle \boldsymbol{C}, D \cup \downarrow \mathit{Supp}(\boldsymbol{B} - \boldsymbol{C})\rangle$$

For function $\mathcal{C}l_{\mathrm{I}}$: recall from Section 5.3 that ideals of $(\mathit{Idl}(X)^{\circledast} \times \mathit{Down}(X), \leq_\times)$ are pairs of ideals of $\mathit{Idl}(X)^{\circledast}$ and of ideals of $\mathit{Down}(X)$. Using the results of Section 7.1, the ideals of $\mathit{Idl}(X)^{\circledast}$ are of the form $\downarrow_\in \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}$ for $(\boldsymbol{\mathcal{B}}, \boldsymbol{D}) \in \mathit{Idl}(\mathit{Idl}(X))^{\circledast} \times \mathit{Down}(\mathit{Idl}(X))$. Let $\langle \downarrow_\in \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}, \boldsymbol{I}\rangle$ be an ideal of $(\mathit{Idl}(X)^{\circledast} \times \mathit{Down}(X), \leq_\times)$:

$$\mathcal{C}l_{\mathrm{I}}(\langle \downarrow_\in \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}, \boldsymbol{I}\rangle) = \langle \downarrow_\in \boldsymbol{\mathcal{B}} \oplus (\boldsymbol{D} \cup \downarrow_\subseteq \{I_1, \ldots, I_n\})^{\circledast}, \boldsymbol{I}\rangle$$

where $I_1 \cup \cdots \cup I_n$ is a decomposition of the downward-closed set $\bigcup \boldsymbol{I}$, and $\downarrow_\subseteq$ here designates the downward-closure of a set of ideals within $\mathit{Down}(\mathit{Idl}(X))$, *i.e.* $\downarrow_\subseteq \boldsymbol{S} = \{I \in \mathit{Idl}(X) \mid \exists J \in \boldsymbol{S}.\ I \subseteq J\}$. It has been shown before that the expression $\bigcup \boldsymbol{I}$ is computable for $\boldsymbol{I} \in \mathit{Idl}(\mathit{Down}(X))$.

Proof. Let $\boldsymbol{B} \in \downarrow_\in \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast} \subseteq \mathit{Idl}(X)^{\circledast}$ and $D \in \boldsymbol{I} \subseteq \mathit{Down}(X)$, so that $\downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$ is an element of $\mathit{Idl}(X^{\circledast})$. Let $\downarrow_\in \boldsymbol{C} \oplus E^{\circledast} \subseteq \downarrow_\in \boldsymbol{B} \oplus D^{\circledast}$ be a smaller ideal. Then by Proposition 7.1.5, $\boldsymbol{C} \setminus \mathit{Down}(D) \subseteq_{\mathrm{emb}} \boldsymbol{B}$ and $E \subseteq D$. Let $I_1 \cup \cdots \cup I_n$ be the ideal decomposition of the downward-closed set $\bigcup \boldsymbol{I}$. Since $\boldsymbol{C}$ is a multiset of ideals, $\boldsymbol{C} \setminus \mathit{Down}(D) = \boldsymbol{C} \setminus \mathit{Idl}(D)$. Indeed, remember that since D is downward-closed, its ideals are exactly the ideals of X that are subsets of D. Moreover, since $D \in \boldsymbol{I}$, then $D \subseteq \bigcup \boldsymbol{I} = I_1 \cup \cdots \cup I_n$. That is, $\mathit{Idl}(D) \subseteq \downarrow_\subseteq \{I_1, \ldots, I_n\}$. Hence, $\boldsymbol{C} \in \downarrow_\in \boldsymbol{\mathcal{B}} \oplus (\boldsymbol{D} \cup \downarrow_\subseteq \{I_1, \ldots, I_n\})^{\circledast}$ and obviously $E \in \boldsymbol{I}$.

The other inclusion is clear. □

Regarding the flattening function: let $\boldsymbol{\mathcal{J}} \in \mathit{Idl}(\mathit{Idl}(X^{\circledast}))$. It is encoded as $\langle \downarrow_\in \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}, \boldsymbol{I}\rangle$ where $\boldsymbol{\mathcal{B}} \in \mathit{Idl}(\mathit{Idl}(X))^{\circledast}$, $\boldsymbol{D} \in \mathit{Down}(\mathit{Idl}(X))$ and $\boldsymbol{I} \in \mathit{Idl}(\mathit{Down}(X))$.

We claim that:

$$\bigcup \boldsymbol{\mathcal{J}} = \downarrow_{\in}\{\!|\bigcup \boldsymbol{I}_1, \ldots, \bigcup \boldsymbol{I}_n|\!\} \oplus (\bigcup \boldsymbol{D} \cup \bigcup \boldsymbol{I})^{\circledast}$$

where $\boldsymbol{\mathcal{B}} = \{\!|\boldsymbol{I}_1, \ldots, \boldsymbol{I}_n|\!\}$, for some $\boldsymbol{I}_1, \ldots, \boldsymbol{I}_n \in \mathit{Idl}(\mathit{Idl}(X))$.

Indeed, let $M \in \bigcup \boldsymbol{J}$, there exists an ideal of $X^{\circledast}$ $\downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast} \in \boldsymbol{J}$ such that $M \in \downarrow_{\in} \boldsymbol{B} \oplus D^{\circledast}$. By definition of $\boldsymbol{\mathcal{J}}$, $D \in \boldsymbol{I}$ and $\boldsymbol{B} \in \downarrow_{in} \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}$. Decompose $M = M_1 + M_2$ with $M_1 \in_{\text{emb}} \boldsymbol{B}$ and $M_2 \in D^{\circledast}$. Firstly, since $D \in \boldsymbol{I}$, $M_2 \in \bigcup \boldsymbol{I}$. Secondly, decompose $\boldsymbol{B} = \boldsymbol{B}_1 + \boldsymbol{B}_2$ with $\boldsymbol{B}_1 \in_{\text{emb}} \boldsymbol{\mathcal{B}}$ and $\boldsymbol{B}_2 \in \boldsymbol{D}^{\circledast}$. Further decompose $M_1 = M_1' + M_1''$ such that $M_1' \in_{\text{emb}} \boldsymbol{B}_1$ and $M_1'' \in_{\text{emb}} \boldsymbol{B}_2$. Composing embeddings that witness $M_1' \in_{\text{emb}} \boldsymbol{B}_1$ and $\boldsymbol{B}_1 \in_{\text{emb}} \boldsymbol{\mathcal{B}}$, it follows that $M_1' \in \{\!|\bigcup \boldsymbol{I}_1, \ldots, \bigcup \boldsymbol{I}_n|\!\}$. Indeed, if an element $x \in M_1'$ belongs to an ideal $I \in \boldsymbol{B}_1$, which itself belongs to an ideal of ideals $\boldsymbol{J} \in \boldsymbol{\mathcal{B}}$, then $x \in \bigcup \boldsymbol{J}$.

Finally, for every $x \in M_1''$, there exists an ideal $I \in \boldsymbol{B}_2$ such that $x \in I$, and since $\boldsymbol{B}_2 \in \boldsymbol{D}^{\circledast}$, $I \in \boldsymbol{D}$. Therefore $x \in \bigcup \boldsymbol{D}$ and $M_1'' \in (\bigcup \boldsymbol{D})^{\circledast}$.

Conversely, let $M = M_1 + M_2 + M_3$ with $M_1 \in_{\text{emb}} \{\!|\bigcup \boldsymbol{I}_1, \ldots, \bigcup \boldsymbol{I}_n|\!\}$ and $M_2 \in (\bigcup \boldsymbol{D})^{\circledast}$ and $M_3 \in (\bigcup \boldsymbol{I})^{\circledast}$. Let $k = |M_1| \leq n$, write $M_1 = \{\!|x_1 \cdots x_k|\!\}$ such that for every $i \in [k]$, $x_i \in \bigcup \boldsymbol{I}_i$. Thus, for each i, there exists $I_i \in \boldsymbol{I}_i$ such that $x_i \in I_i$. Define I_j to be an arbitrary ideal of $\boldsymbol{I}_j$ for $j \in [k+1, n]$, and define $\boldsymbol{B} = \{\!|I_1, \ldots, I_n|\!\}$. We have $M_1 \in_{\text{emb}} \boldsymbol{B} \in_{\text{emb}} \boldsymbol{\mathcal{B}}$.

Moreover, let $M_2 = \{\!|y_1 \cdots y_\ell|\!\}$ and $\boldsymbol{B}' = \{\!|\downarrow y_1, \ldots, \downarrow y_\ell|\!\} \in (\mathit{Idl}(X))^{\circledast}$. Since $M_2 \in (\bigcup \boldsymbol{D})^{\circledast}$, $\boldsymbol{B}' \in \boldsymbol{D}^{\circledast}$, and $\boldsymbol{B} + \boldsymbol{B}' \in \downarrow_{\in} \boldsymbol{\mathcal{B}} \oplus \boldsymbol{D}^{\circledast}$. Finally, we can similarly prove that there exists $D \in \boldsymbol{I} \subseteq \mathit{Down}(X)$ such that $M_3 \in D^{\circledast}$, ultimately proving that $M \in \bigcup \boldsymbol{\mathcal{J}}$.

Finite Powerset with Hoare Quasi-Ordering: In this case, $\mathit{Idl}(\mathcal{P}_f(X), \subseteq)$ is isomorphic to $(\mathit{Down}(X), \subseteq)$, which has been shown to be ideally effective in Proposition 9.4.2. The computability of the flattening function has also been proved there.

□

Unfortunately, we did not finish our investigation in time to include all the results in this manuscript, but we have recently obtained that:

- Under extra assumptions, $(X, \leq')$ is Idl^2-effective when $(X, \leq)$ is, where $\leq \subseteq \leq$.
- These extra assumptions are met in the case of $(X^*, \leq_{\text{cj}})$ which is thus Idl^2-effective.
- The domination ordering on finite multisets is also Idl^2-effective.

Remark 9.4.5. *As the reader may have noticed, constructions that "commute" with the ideal constructions are particularly easy to show Idl^2-effective. For instance, for the cartesian product, $\mathit{Idl}(X_1 \times X_2) = \mathit{Idl}(X_1) \times \mathit{Idl}(X_2)$ and therefore, the Idl^2-effectiveness comes for free with the ideal-effectiveness.*

In the case of sequences, we have seen that ideals of X^* are some kind of "higher" sequences as well, but the ordering on these sequences is not the classical embedding ordering. Therefore, we introduced a quasi-ordering $\leq_{\text{st}}^{A}$ on sequences that generalizes *the Higman quasi-ordering, and that almost commutes with the ideal construction. In particuar, Chapter 6 should have presented $(X^*, \leq_{\rm st}^{A})$ first, and proved its ideal effectiveness. Then, the ideal effectiveness of $(X^*, \leq_*)$ and $(X^*, \leq_{\rm st})$ would have been obtained as a corollary.*

The answer to the Idl^2-effectiveness of extensions ($\leq \subseteq \leq'$, see Section 4.1) follows a similar path: the ideals of $(X, \leq')$ is not an extension of the ideals of $(X, \leq)$, essentially because the support of the QO is not the same: $Idl(X, \leq) \neq Idl(X, \leq')$. What we found to be the good notion that generalizes extensions is the setting where there exists a surjective and monotone function $f : (X, \leq_X) \to (Y, \leq_Y)$. In this case, $(X, \leq_X)$ is ideally effective, and functions $\mathcal{C}l_{\rm I}$ and $\mathcal{C}l_{\rm F}$ (similar definitions) are computable, then $(Y, \leq_Y)$ is ideally effective. Moreover, it is now possible to show that this setting is preserved under the ideal construction: the existence of f always imply the existence of a surjective and monotone map $g : Idl(X, \leq_X) \to Idl(Y, \leq_Y)$. Thanks to this more general notion, we were able to show that $(X^*, \leq_{\rm cj})$ is Idl^2-effective.

9.5 Perspectives

In conclusion, we have provided an effective algebra of ideally effective ω^2-WQOs, whose set-theoretic operations in these ω^2-WQOs can be automatically computed. Most of the commonly used WQOs fall in this algebra, but this inductive approach becomes particularly handy for WQOs that consist of a high number of iterations of these constructions. This is for instance the case of the quasi-ordering used for *priority channel systems* [47], which essentially is the n-th iteration of the Higman's extension of the one element WQO, for a fixed $n \in \mathbb{N}$.

Note that this algebra is not closed under taking the infinite sequences. Indeed, this construction is not Idl^2-effective as it does not even preserve ω^2-WQOs. However, given any quasi-ordering in this algebra, the set of infinite sequences over this WQO is ideally effective, and procedures for set-theoretic operations can be computed. The same can be said for taking the powerset $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$ of a quasi-ordering in this algebra. This has applications in verification: some algorithms to verify WSTS [48, 49] rely on $(\mathcal{P}(X), \sqsubseteq_{\mathcal{H}})$ being an effective WQO. Also in [29], the authors define the *completion* of a WSTS which is a transition system whose states are the ideals of the original set of states. For this system to be well-structured, we need that the original WQO is ω^2-WQO, and to apply *e.g.* forward analysis, it is crucial that $Idl(X)$ be an ideally effective WQO.

Extending the Algebra One of the most natural extension of our work would be to add more classical constructions to our algebra, the most relevant being trees and graphs. In these two cases, the objects at hand are more complex, and therefore so are the ideals, and a fortiori operations manipulating them. Already characterizing the structure of the ideals is difficult, and finding convenient representations becomes tough as well. Moreover, there are several class of trees to consider, each of which can be quasi-ordered with several variants (bounded/unbounded width, bounded/unbounded height, ranked/unranked trees, etc.). The case of graph is no simpler: already the minor relation is not simple to manipulate, and in practice many variants of this quasi-ordering are used, that are sometimes only WQOs on certain classes of graphs.

In conclusion, a complete investigation of these two cases would be long and difficult, and the results would be quite technical. In the case of trees, some cases have been shown effective (in some sense close to ours) in [13] and [30] for instance. The second case comes with a clear motivation.

Infinite Sequences. If we want the infinite sequences construction to be part of our algebra, we need a class of WQOs that is preserved by this construction. Section 9.1 shows that this is the same as being preserved by the construction $Down$. From Proposition 9.3.4 we deduce that the smallest such class is that of α-WQOs for $\alpha < \omega^\omega$.

This motivates the following inductive definition:

Definition 9.5.1. *Given $n \in \mathbb{N}$, $(X, \leq)$ is Idl^n-WQO if:*

- *$(X, \leq)$ is ω^n-WQO,*
- *$(X, \leq)$ is an ideally effective WQO,*
- *$(Idl(X), \subseteq)$ is Idl^{n-1}-effective,*
- *The representation of ideals of X coincides in the two lines above.*
- *The flattening function from $Idl(Idl(X))$ to $Idl(X)$ is computable.*

$(X, \leq)$ is Idl^ω-effective if it is Idl^n-effective for every $n \in \mathbb{N}$.

Note that from Proposition 9.3.4, it is simple to derive that if $(X, \leq)$ is ω^n-WQO, then $Idl^n(X)$ is a WQO, where $Idl^n(X)$ is the n-th composition of the ideal completion of X. Note that X ω^ω-WQO is not equivalent to X being ω^n-WQO for every n. Indeed, Proposition 9.3.4 shows that if X is ω^n-WQO for every n, then so is $Idl(X)$. However, $Idl(X)$ is ω^ω-WQO if and only if X is $\omega^{\omega+1}$-WQO. Therefore, the two conditions are not equivalent since it is proved in [45] that α-WQO and β-WQO are two different notions when α and β are two distinct indecomposable countable ordinals.

However, for our purpose, being ω^n-WQO for every $n \in \mathbb{N}$ seems sufficient: it allows any finite number of application of the infinite sequences construction to a WQO of our algebra, provided all of our constructions are Idl^n-effective (similar definition)

Disjoint and lexicographic sums, and cartesian products are obviously Idl^ω effective. But this is less clear for finite sequences for instance, since $Idl(X^*)$ is not directly expressible with our constructions (we had to use an extension).

A promising first step would be to generalize Section 4.1 to Idl^n-effective WQOs. Does assuming the computability of $\mathcal{C}l_{\mathrm{I}}$ and $\mathcal{C}l_{\mathrm{F}}$ at the first level is enough to show that an extension is still Idl^n-effective, or do we need such functions for each level $Idl^m(X)$ for $m \leq n$?

Ideally Effective BQOs. What about a notion of effectiveness for BQOs? The characterization X ω^n-WQO if and only if $Idl^n(X)$ WQO is convenient to inductively define Idl^n-effectiveness. But does it generalizes further ? There is a classical definition of the transfinitely iterated powerset $\mathcal{P}^\alpha(X)$ in BQO theory, which can be adapted to $Idl^\alpha(X)$, and it is well known that X is BQO if and only if $\mathcal{P}^{\omega_1}(X)$ is WQO. However, we did not find any layered version of this theorem, that is: X $\omega^{\alpha+1}$-WQO if and only if $\mathcal{P}^\alpha(X)$ WQO. With such a property, we could for instance define Idl^α-effectiveness as $Idl^\beta(X)$ being ideally effective for every $\beta \leq \alpha$. It would probably also be necessary to assume that representations used for each $Idl^\beta(X)$ are compatible, in the sense that the representation for ideals of $Idl^\beta(X)$ is the same as the representation for the elements of $Idl^{\beta+1}(X)$; and to have some uniformity assumptions of the form: the function that to $\beta \in \boldsymbol{\alpha}$ associates a full presentation of $Idl^\beta(X)$ is computable.

Minimality of the Definition Is Definition 9.4.1 minimal ? Does there exists a ω^2-WQO $(X, \leq)$ such that $(Idl(X), \subseteq)$ is not ideally effective ? A starting point would be to investigate the domination ordering on finite multisets or the lexicographic product, since we were unable to prove these constructions to be Idl^2-effective.

Part II

First-Order Logic over an Ideally Effective WQOs

Joint work with Ph. Schnoebelen and G. Zetzsche

Results on the expressiveness and decidability of first-order theories, and in particular first-order theories over some fixed structure abound. One of the simplest structures one can think of are quasi-ordered sets, and indeed many such structures have been studied in the last decades (see [50, 51, 52, 53, 54, 55, 56] and references therein). In the following chapters, we investigate first-order logics over well quasi-ordered sets. More precisely, given a WQO $(X, \leq)$, we consider the first-order logic with $\leq$ as the only predicate, denoted $\mathsf{FO}(X, \leq)$, where $\leq$ is interpreted as the quasi-ordering $\leq$ over X. We can for instance express the following properties:

- $\forall x, y, z.\ x \leq y \wedge y \leq z \to x \leq z$. This expresses that $\leq$ is transitive, which is true for any QO $(X, \leq)$
- $\forall x, y.\ \exists u.\ x \leq u \wedge y \leq u \wedge [\forall z.\ (x \leq z \wedge y \leq z) \to u \leq z]$. This expresses that $(X, \leq)$ is a semi-lattice. This formula is for instance satisfied on the structure $(\mathbb{N}^k, \leq_\times)$ but not on $(A^*, \leq_*)$.
- $\exists x, y, z, u, v, w.\ x \leq u \wedge x \leq v \wedge x \not\leq w \wedge y \leq u \wedge y \not\leq v \wedge y \leq w \wedge z \not\leq u \wedge z \leq v \wedge z \leq w \wedge x \perp y \wedge x \perp z \wedge y \perp z \wedge u \perp v \wedge u \perp w \wedge v \perp w$, where $x \perp y$ is an abbreviation for $x \not\leq y \wedge y \not\leq x$. This formula expresses that some finite ordering embeds into $(X, \leq)$. For instance, this particular finite ordering does not embed into $\mathbb{N}^2$, but does embed into $\mathbb{N}^k$ for $k \geq 3$.

We also consider formulas that include constants from X, that is we consider the first-order theory with $\leq$ as the only predicate, and a (potentially infinite) set of constants $\mathtt{X}$, interpreted over X. This logic will be denoted $\mathsf{FO}(X, \leq, \mathtt{X})$. For instance, the formula

$$\forall x.\ (x \geq \langle 2, 3\rangle \wedge x \geq \langle 3, 2\rangle) \leftrightarrow x \geq \langle 3, 3\rangle$$

expresses that over $\mathbb{N}^2$, $\uparrow\langle 2, 3\rangle \cap \uparrow\langle 3, 2\rangle = \uparrow\langle 3, 3\rangle$.

More generally, we can represent any upward-closed set $U = \bigcup_{i=1}^n \uparrow x_i$ as a formula $\varphi_U(x) = \bigvee_{i=1}^n x \geq \mathtt{x}_i$ with one free variable x, such that for any element $\mathtt{x}$ of X, $\mathtt{x} \in U$ if and only if $X, x \mapsto \mathtt{x} \models \varphi(x)$. Therefore, the logic provides a representation for upward-closed sets, and also for downward-closed sets using the *excluded minor* representation. Moreover, the logic can express that a "set" (represented as a formula with one free variable) is closed, or directed:

$$\forall x, y.\ \varphi(x) \wedge y \geq x \Rightarrow \varphi(y)$$
$$\forall x, y. \varphi(x) \wedge \varphi(y) \Rightarrow \exists z.\ \varphi(z) \wedge z \geq x \wedge z \geq y$$

As a result, if $\mathsf{FO}(X, \leq, \mathtt{X})$ is decidable, we can compute unions, intersections and complements of closed subsets. In the following chapters, we investigate connections between this representation of closed subsets and the notion of effectiveness introduced in the first part. Of course, the decidability of the full logic $\mathsf{FO}(X, \leq, \mathtt{X})$ seems to be a much stronger property, and this is confirmed by the undecidability of $\Sigma_2(A^*, \leq_*)$, proved in [55] already for A a two-elements alphabet. In the following chapters, we thus mostly study sub-fragments of the existential fragment $\Sigma_1(X, \leq, \mathtt{X})$. The existential fragment of a structure remains an important piece of the logic, notably due to the success of SMT solvers over the past decades. It corresponds to constraint solving, which has many applications in theorem proving and rewriting theory for instance [50].

In Chapter 10, we prove that the *positive existential fragment* of $\mathsf{FO}(X, \leq, \mathtt{X})$ is decidable for any ideally effective WQO $(X, \leq)$. In [50], it is shown that the positive existential fragment of first-order logic interpreted over terms on a finite signature, and where the only predicate $\leq$ is interpreted as tree embedding, is decidable. In these terms, we extend this result to the case where the set of terms is generated by an infinite signature, but with no symbols of arity greater than 0 (*i.e.* only constants). In this case, the ordering on terms (trees) coincides with the ordering over the elements.

In Chapter 11, we show that our result cannot be extended to the full existential fragment, since $\Sigma_1(A^*, \leq_*, \mathtt{A}^*)$ is already undecidable for a two-symbols alphabet A, while $(A^*, \leq_*)$ is ideally effective (Section 6.1).

The first-order structure of words (over a finite alphabet) is of particular importance in computer science, and it has been studied for various orderings, although the most studied structure on words probably is $(A^*, \cdot, =)$ where $\cdot$ is interpreted as concatenation. Its Σ_2 fragment is undecidable [57, 58], but its existential fragment has been shown decidable by Makanin [59], and has been intensively studied since, notably because its exact complexity is still an open problem [60, 61]. In [54], first-order logics over A^* with several orderings are considered. It is in particular shown that $\Sigma_3(A^*, \leq_*)$ is undecidable. This result is improved in [55]: $\Sigma_2(A^*, \leq_*)$ is undecidable. Note that in the case of A^*, it is possible to define constants (up to permutation of the letters of A) in the Σ_2 fragment, as shown in [55]. However, without constants, $\Sigma_1(A^*, \leq_*)$ is decidable (this is discussed in Section 10.3). In Section 11.1, we close the gap, showing that in the presence of constants, $\Sigma_1(A^*, \leq_*, \mathtt{A}^*)$ is undecidable.

Chapter 11 is based on the article [62]: only parts of the article are rewritten here. A brief overview of the results of [62] that are not presented in this thesis is given in conclusion.

Chapter 10

Constraints Solving on an Ideally Effective WQO

10.1 Definitions

In this section, we define formally all the notions discussed in the previous introduction that will be used in the subsequent sections of this chapter. The main object is the first-order logic over the structure $(X, \leq)$ with constants, denoted $\mathsf{FO}(X, \leq, \mathtt{X})$.

Syntax. The syntax of this logic consists of all first-order formulas over the signature with predicate $\leq$ and a constant symbol $\mathtt{u}$ for each $u \in X$. We use the font $\mathtt{u}$ to distinguish constants from variables in the formulas. These formulas are generated by the following grammars:

- Terms: $t ::= x \mid \mathtt{u}$
  where $\mathtt{u} \in \mathtt{X}$ and $x \in \mathrm{Var}$, a countably infinite set of variables.

- Formulas: $\varphi ::= t \leq t \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\, \varphi$

The notation $\mathsf{FO}(X, \leq)$ denotes the first-order logic over X without constant symbols, *i.e.* where terms are restricted to be variables from Var.

Semantic. We interpret these formulas over a WQO $(X, \leq)$: a *valuation* is a function from Var to X. A formula φ is said to be *satisfied* by a valuation $V : \mathrm{Var} \to X$, denoted by $X, V \models \varphi$, if:

$$X, V \models t_1 \leq t_2 \quad \overset{\text{def}}{\Leftrightarrow} V(t_1) \leq V(t_2), \text{ where } V \text{ is extended to constants by } V(\mathtt{u}) \overset{\text{def}}{=} u$$

$$X, V \models \neg\varphi \quad \overset{\text{def}}{\Leftrightarrow} X, V \models \varphi$$

$$X, V \models \varphi_1 \wedge \varphi_2 \quad \overset{\text{def}}{\Leftrightarrow} X, V \models \varphi_1 \text{ and } X, V \models \varphi_2$$

$$X, V \models \varphi_1 \vee \varphi_2 \quad \overset{\text{def}}{\Leftrightarrow} X, V \models \varphi_1 \text{ or } X, V \models \varphi_2$$

$$X, V \models \exists x.\, \varphi \quad \overset{\text{def}}{\Leftrightarrow} \text{ there exists } z \in X \text{ such that } X, V \uplus (x \mapsto z) \models \varphi$$

where $V \uplus (x \mapsto z)$ denotes the valuation such that for any $y \in \mathrm{Var} \setminus \{x\}$, $V \uplus (x \mapsto z)(y) = V(y)$ and $V \uplus (x \mapsto z)(x) = z$.

Truth Problems. A formula is said to be:

- *satisfiable* if there exists a valuation $V : \mathrm{Var} \to X$ such that $X, V \models \varphi$;
- *valid* if for any valuation $V : \mathrm{Var} \to X$, $X, V \models \varphi$

The set of *free variables* of a formula φ will be denoted $\mathsf{fv}(\varphi)$. We sometimes write $\varphi(x_1, \dots, x_n)$ to emphasize that $\mathsf{fv}(\varphi) = \{x_1, \dots, x_n\}$ where the x_i's are distinct. When such an enumeration $x_1, \dots, x_n$ of $\mathsf{fv}(\varphi)$ is understood, we denote the *solutions* of φ by $[\![\varphi]\!]$, defined by:

$$[\![\varphi]\!] \overset{\text{def}}{=} \{(z_1, \dots, z_n) \in X^n \mid X, \biguplus_{i \in [n]} (x_i \mapsto z_i) \models \varphi\}$$

Note that with this definition, a formula φ is satisfiable if and only if $[\![\varphi]\!] \neq \emptyset$; and valid if and only if $[\![\varphi]\!] = X^n$. As a consequence, observe that a formula is satisfiable if and only if its negation is not valid.

A formula is said to be *closed* if it has no free variables. In this case, satisfiability and validity coincide (the only valuation being the empty valuation), and the two notions will be used interchangeably.

In the remainder of Part II, we study satisfiability and validity over fragments of $(X, \leq)$: given a *fragment* (*i.e.* subset) $\mathcal{F}$ of $\mathsf{FO}(X, \leq, \mathtt{X})$, define the following problems,

<u>SATISFIABILITY OF THE $\mathcal{F}$ FRAGMENT</u>
INPUT: A first-order formula $\varphi \in \mathcal{F}$
QUESTION: Is φ satisfiable ?

<u>VALIDITY OF THE $\mathcal{F}$ FRAGMENT</u>
INPUT: A first-order formula $\varphi \in \mathcal{F}$
QUESTION: Is φ valid ?

If $\mathcal{F}$ consists of closed formulas only (and it will always be the case), then the two problems coincide, and we will say that $\mathcal{F}$ is decidable (*resp.* undecidable) to express that both problems are decidable (*resp.* undecidable).

Equivalence and Normal Forms Two formulas φ and ψ are said to be *equivalent*, denoted $\varphi \Leftrightarrow \psi$, if $[\![\varphi]\!] = [\![\psi]\!]$.

We assume the reader is familiar with the following normal forms:

- Alpha-renaming: if φ is quantifier-free and $y \notin \mathsf{fv}(\varphi)$, then formulas $\exists x.\ \varphi$ and $\exists y.\ \varphi[x \leftarrow y]$ are equivalent, where $\varphi[x \leftarrow y]$ denotes the syntactical substitution of variable y to variable x in φ. In particular, we can always assume that a variable is not quantified twice in a first-order formula.

- Prenex normal form: any formula φ is equivalent to a formula in *prenex normal form*, that is with all quantifiers in front; *i.e* of the form $Qx_1.\ Qx_2 \dots Qx_n.\ \psi$ where ψ is quantifier-free and $Q \in \{\exists, \forall\}$. Given a formula, one can compute an equivalent formula in prenex normal form.

- Finally, quantifier-free formulas can be put in *disjunctive (*resp. *conjunctive) normal form*, that is as a disjunction of conjunctions of litterals, where a *litteral* is either an atomic formula or its negation.

Since the problems we study are closed under equivalence, we can always assume that the input formulas of our problems are given in such forms.

Fragments. The main fragments we are interested in are fragments where the quantifier alternation is controlled. Let Σ_i be the set of closed formulas in prenex normal form that start with a certain number of consecutive existential quantifiers, then a certain number of consecutive universal quantifiers, and so on alternating at most i times between consecutive blocks of each type of quantifiers. The fragment Π_i is defined analogously, but we ask that formulas start with universal quantifiers. By convention, $\Sigma_0 = \Pi_0$ is the set of quantifier-free formulas.

The fragment Σ_i of $\mathsf{FO}(X, \leq, \mathtt{X})$ will be denoted $\Sigma_i(X, \leq, \mathtt{X})$.

Syntactic Sugar. In the subsequent chapters, we will use the usual syntactic abbreviations for logical operators (notably for the universal quantification $\forall$), as well as the following shortcuts:

- Not smaller: $x \not\leq y \overset{\text{def}}{\Leftrightarrow} \neg(x \leq y)$.

- Order equivalence: $x \equiv y \overset{\text{def}}{\Leftrightarrow} x \leq y \wedge y \leq x$. The symbol $\equiv$ will be denoted $=$ if $(X, \leq)$ is anti-symmetric.

- Incomparability: $x \perp y \overset{\text{def}}{\Leftrightarrow} x \not\leq y \wedge y \not\leq x$.

- Membership in an upward-closed set: if $U = \bigcup_i \uparrow u_i$, $x \in U \overset{\text{def}}{\Leftrightarrow} \bigvee_i x \geq \mathtt{u}_i$.

- Membership in a downward-closed set: if D is downward-closed, its complement $\complement D = X \setminus D$ is upward-closed, and the previous abbreviation can be used: $x \in D \overset{\text{def}}{\Leftrightarrow} \neg(x \in \complement D)$.

10.2 Positive Existential Fragment

We begin our investigation with a very simple fragment of $\mathsf{FO}(X, \leq, \mathtt{X})$, which we call the *extended positive existential fragment*. As its name indicates, it is a sub-fragment of the more common *existential fragment* $\Sigma_1(X, \leq, \mathtt{X})$. In this section, we show the decidability of the extended positive existential fragment, under the assumption that $(X, \leq)$ is an ideally effective WQO.

Definition 10.2.1. *The* extended positive existential fragment *of* $\mathsf{FO}(X, \leq, \mathtt{X})$ *is defined by the following grammar:*

$$\varphi ::= \mathtt{u} \leq x \mid \mathtt{u} \not\leq x \mid x \leq \mathtt{u} \mid x \not\leq \mathtt{u} \mid x \leq y \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\, \varphi$$

where $c \in X$ *and* $x, y \in \mathrm{Var}$.

We call it extended due to the presence of negation in some atomic formulas. Observe that it suffices to add the predicate "$x \not\leq y$" to this grammar to generate the full existential fragment of $\mathsf{FO}(X, \leq, \mathtt{X})$.

Theorem 10.2.2. *If* $(X, \leq)$ *is an ideally effective WQO, then the extended positive existential fragment of* $\mathsf{FO}(X, \leq, \mathtt{X})$ *is decidable.*

Our first observation is that the four atomic formulas $x \bowtie \mathtt{u}$ (for $x \in \mathrm{Var}$ a variable, $\mathtt{u} \in \mathtt{X}$ a constant and $\bowtie \in \{\leq, \geq, \not\leq, \not\geq\}$) can be reformulated as $x \in U$ for some upward-closed set $U \subseteq X$ or as $x \in D$ for some downward-closed set $D \subseteq X$. Indeed:

$$\mathtt{u} \leq x \Leftrightarrow x \in {\uparrow} u \qquad \mathtt{u} \not\leq x \Leftrightarrow x \in \neg{\uparrow} u$$
$$x \leq \mathtt{u} \Leftrightarrow x \in {\downarrow} u \qquad x \not\leq \mathtt{u} \Leftrightarrow x \in \neg{\downarrow} u$$

Since $(X, \leq)$ is an ideally effective WQO, we can compute these sets (which actually are principal ideals and filters). We push this approach beyond atomic formulas: given a closed formula of the extended positive existential fragment, we can compute its prenex normal form, and put the quantifier-free part of the formula in Disjunctive Normal Form. We obtain a formula $\varphi = \exists x_1 \cdots \exists x_k.\ \bigvee_{i=1}^{p} \bigwedge_{j=1}^{q_i} \varphi_{i,j}$ where the $\varphi_{i,j}$ are atomic. Moreover, since $(X, \leq)$ is ideally effective, we can compute intersections of closed subsets. Thus for each $i \in [p]$ and each $j \in [q_i]$, there exist formulas ψ_1 and ψ_2 such that $\bigwedge_{j=1}^{q_i} \varphi_{i,j}$ is equivalent to $\psi_1 \wedge \psi_2$, ψ_1 is a conjunction of exactly one constraint of the form $x \in U_x \cap D_x$ per $x \in \mathrm{Var}(\varphi)$, for some upward-closed set U_x and some downward-closed set D_x, and ψ_2 is a conjunction of constraints of the form $x \leq y$ for some $x, y \in \mathrm{Var}(\varphi)$. In the end, satisfiability of a conjunction $\bigwedge_{j=1}^{q_i} \varphi_{i,j}$ reduces to the following problem we call *Partial-Embeddability under constraints*:

PARTIAL-EMBEDDABILITY UNDER CONSTRAINTS

INPUT: A finite quasi-ordering $(V, \leq)$, a collection of upward-closed sets $(U_v)_{v \in V}$ of X, and a collection of downward-closed sets $(D_v)_{v \in V}$ of X.

QUESTION: Does there exist a monotone $f : V \to X$ such that $\forall v \in V.\ f(v) \in U_v \cap D_v$?

Indeed, we take $V = \bigcup_{j=1}^{q_i} \mathsf{fv}(\varphi_{i,j})$; ψ_2 induces a quasi-ordering on V, that is, $x \leq y$ in V if and only if $x \leq y$ is a conjunct of ψ_2. Finally, ψ_1 gives a collection of upward-closed sets $(U_v)_{v \in V}$ and a collection of downward-closed sets $(D_v)_{v \in V}$. Subsequently, we show that this problem is decidable, which proves the decidability of the extended positive existential fragment since it then suffices to solve an instance of the Partial-Embeddability under constraints problem for each $\bigwedge_{j=1}^{q_i} \varphi_{i,j}$.

Lemma 10.2.3. *The Partial-Embeddability under constraints problem is decidable.*

Proof. The algorithm to solve the Partial-Embeddability under constraints problem is based on the following observation: given $v_1, v_2 \in V$ such that $v_1 \leq v_2$, the function f we seek to define must satisfy the following conditions:

1. $f(v_1) \leq f(v_2)$ (f is monotone),
2. $f(v_1) \in U_{v_1} \cap D_{v_1}$,
3. $f(v_2) \in U_{v_2} \cap D_{v_2}$.

But, 1 and 3 together imply that $f(v_1) \in D_{v_2}$, and 1 and 2 together imply that $f(v_2) \in U_{v_1}$. That is to say, our instance is equi-satisfiable with the same instance where D_{v_1} is replaced by $D_{v_1} \cap D_{v_2}$ and U_{v_2} is replaced by $U_{v_2} \cap U_{v_1}$.

This motivates the following definition: an instance of the Partial-Embeddability problem under constraints is said to be *resolved* if:

- R1: for every $v_1, v_2 \in V$, $v_1 \leq v_2 \Rightarrow U_{v_2} \subseteq U_{v_1}$,
- R2: for every $v \in V$, U_v is a filter,
- R3: for every $v_1, v_2 \in V$, $v_1 \leq v_2 \Rightarrow D_{v_1} \subseteq D_{v_2}$,
- R4: for every $v \in V$, D_v is an ideal.

Observe that the satisfiability status of a resolved instance $\mathcal{I} = (V, \leq, (\uparrow x_v)_{v \in V}, (I_v)_{v \in V})$ is immediate, hence the name:

$$\begin{aligned} \mathcal{I} \text{ is a yes-instance } &\Leftrightarrow \forall v \in V.\ \uparrow x_v \cap I_v \neq \emptyset \\ &\Leftrightarrow \forall v \in V.\ x_v \in I_v \end{aligned}$$

This last condition is decidable (axiom (IM)). The left-to-right direction of the first equivalence is trivial. For the left-to-right direction of the second equivalence: if there exists some $y \in \uparrow x_v \cap I_v$, then in particular $x_v \leq y \in I_v$, thus $x_v \in I_v$ since I_v is downward-closed. Finally, if $x_v \in I_v$ for every $v \in V$, then the mapping $f(v) = x_v$ is a solution to $\mathcal{I}$. Indeed:

- for $v \in V$, $x_v \in \uparrow x_v \cap I_v$.
- given $v_1, v_2 \in V$, if $v_1 \leq v_2$ then $\uparrow x_{v_2} \subseteq \uparrow x_{v_1}$,
  i.e. $x_{v_1} = f(v_1) \leq x_{v_2} = f(v_2)$.

Now to conclude the proof, we show how to reduce the satisfiability of any instance to the satisfiability of a finite number of resolved instances. This process is essentially the one described in our first observation: if an instance is not resolved, then there exists a pair $(v_1, v_2) \in V^2$ such that $v_1 \leq v_2$ and either $U_{v_2} \not\subseteq U_{v_1}$ or $D_{v_1} \not\subseteq D_{v_2}$, or both. In this case, we can update U_{v_2} to $U_{v_2} \cap U_{v_1} \subseteq U_{v_1}$ (respectively D_{v_1} to $D_{v_1} \cap D_{v_2} \subseteq D_{v_2}$). In this instance, the pair (v_1, v_2) no longer violates conditions R1 and R3 of the definition of resolved (we will deal with conditions R2 and R4 later).

However, this new instance might not be resolved because of another pair (v_3, v_4), and it might be the case that v_3 or v_4 is actually equal to v_1 or v_2. Then, performing the same update for (v_3, v_4) might break again the property $U_{v_2} \subseteq U_{v_1} \wedge D_{v_1} \subseteq D_{v_2}$. Therefore we must be careful in which order the updates are performed.

Observe that the update for a pair (v_1, v_2) does not change D_{v_2} and U_{v_1}. We thus have to consider the upward-closed sets and the downward-closed sets independently, and update upward-closed sets following a non-decreasing order with respect to $(V, \leq)$, while updating downward-closed sets following a non-increasing order with respect to $(V, \leq)$.

Besides, to ensure conditions R2 and R4, we use the decomposition of upward-closed sets (*resp.* downward-closed sets) as a union of filters (*resp.* ideals). If $U = \bigcup_i F_i$ and $D = \bigcup_j I_j$, then $x \in U \cap D$ is equivalent to $\bigvee_{i,j} x \in F_i \cap I_j$. Because of this disjunction, one instance will be reduced into several, which will all be further reduced until all the instances are resolved. The original instance is satisfiable if and only if one of the resulting resolved instances is satisfiable.

Formally, given an unresolved instance $(V, \leq, (U_v)_{v \in V}, (D_v)_{v \in V})$, let $n = |V|$ and label the elements of V as $v_1, \ldots, v_n$ such that $\forall i, j \in [n], i < j \Rightarrow v_i \not\geq v_j$. In other words, $v_1 \preccurlyeq v_2 \preccurlyeq \cdots \preccurlyeq v_n$ is a linearization of $\leq$. We then run the following algorithm:

1. Let L be a singleton list containing the original instance.
2. Let L' be an empty list, that will be used as a temporary variable.
3. for $j = 1$ to n:
    (a) for each instance $\mathcal{I}$ in L:
        i. Update $U_{v_j} := U_{v_j} \cap \bigcap_{i<j} U_{v_i}$ in $\mathcal{I}$.
        ii. For each $x \in \min(U_{v_j})$, add a version of $\mathcal{I}$ updated with $U_{v_j} := \uparrow x$ to L'.
    (b) $L := L'$.
4. for $i = n$ down to 1:
    (a) for each instance $\mathcal{I}$ in L:
        i. Update $D_{v_i} := D_{v_i} \cap \bigcap_{i<j} D_{v_j}$ in $\mathcal{I}$.
        ii. For each ideal I in the canonical ideal decomposition of D_{v_i}, add a version of $\mathcal{I}$ updated with $D_{v_i} := I$ to L'.
    (b) $L := L'$.

5. Return L.

As observed at the beginning of this proof, steps 3(a)i and 4(a)i preserve satisfiability, and steps 3(a)ii and 4(a)ii simply transform unions into disjunction over multiple instances. The first loop (at line 3) updates upward-closed sets, while the second loop (at line 4) updates downward-closed sets. After the first loop is executed for index j, the following holds for every instance of L:
$\forall v \in V.\ v \leq v_j \Rightarrow U_v \subseteq U_{v_j}$, and the upward-closed sets U_v for $v \leq v_j$ will no longer be modified. Moreover, U_{v_j} is a filter, and will no longer be modified. Thus conditions R1 and R2 hold at the end of the algorithm. A similar invariant holds for the second loop. Hence, at the end of the algorithm, all instances in L are resolved, and the original instance is satisfiable if and only if one of the resolved instances of L is. $\square$

10.3 Full Existential Fragment

In the previous section, we have established the decidability of a sub-fragment of the existential fragment of $\mathsf{FO}(X, \leq, \mathtt{X})$. As observed earlier, it suffices to add the predicate $x \not\leq y$ to the grammar of Definition 10.2.1 to obtain a grammar for the full existential fragment of $\mathsf{FO}(X, \leq, \mathtt{X})$ (this grammar produces existential formulas whose negations have been pushed down to the atoms).

The same approach we used for the extended positive existential fragment can be applied to the full existential fragment:

- push the negations of φ in,
- convert predicates $x \bowtie c$ into constraints $x \in U$ and $x \in D$,
- put in prenex normal form, with the quantifier-free part in DNF.

We obtain a formula

$$\varphi = \exists x_1 \cdots \exists x_k.\ \bigvee_{i=1}^{p} \left(\bigwedge_{x \in \mathrm{Var}} x \in U_x^{i,j} \cap D_x^{i,j} \right) \wedge \psi_{i,j}$$

for some collections of upward-closed sets $(U_x)_{x \in \mathrm{Var}}$ and downward-closed sets $(D_x)_{x \in \mathrm{Var}}$. However this time, formulas $\psi_{i,j}$ are conjuncts of predicates of the form $x \leq y$ for some $x, y \in \mathrm{Var}$ (as before) but also of the form $x \not\leq y$. Thus, if we interpret $\psi_{i,j}$ as an quasi-ordering on Var, some monotone mapping from Var to X might actually not satisfy $\psi_{i,j}$. We need the stronger notion of *embedding*: $x \leq y$ if and only if $f(x) \leq f(y)$. We obtain a reduction to the following problem:

<u>EMBEDDABILITY UNDER CONSTRAINTS</u>

INPUT: A finite quasi-ordering $(V, \leq)$, and two collections $(U_v)_{v \in V}$ and $(D_v)_{v \in V}$ of upward-closed and downward-closed sets of X respectively.

QUESTION: Does there exists an <u>embedding</u> $f : V \to X$ such that $\forall v \in V.\ f(v) \in U_v \cap D_v$?

Note that since $\psi_{i,j}$ might not fully define an quasi-ordering on Var (*i.e.* ψ does not enforce exactly one among $x < y$, $x = y$ or $x \perp y$), we have to make an additional disjunction in φ over all finite quasi-orderings satisfying $\psi_{i,j}$ before reducing to this problem.

Replacing the monotone mapping by an embedding induces some changes: we say that an instance $(V, \leq, (U_v)_{v\in V}, (D_v)_{v\in V})$ of the Embeddability under constraints problem is *resolved* if:

1. for every $v_1, v_2 \in V$, $v_1 \leq v_2 \Rightarrow U_{v_2} \subseteq U_{v_1}$,
2. for every $v \in V$, U_v is a filter,
3. for every $v_1, v_2 \in V$, $v_1 \leq v_2 \Rightarrow D_{v_1} \subseteq D_{v_2}$,
4. for every $v \in V$, D_v is an ideal.
5. for every $v_1, v_2 \in V$, $v_1 \not\leq v_2 \Rightarrow U_{v_2} \not\subseteq U_{v_1}$

Once again, a resolved instance $\mathcal{I} = (V, \leq, (\uparrow x_v)_{v\in V}, (I_v)_{v\in V})$ can be easily solved:

$$\begin{aligned}\mathcal{I} \text{ is a yes-instance if and only if } & \forall v \in V.\ \uparrow x_v \cap I_v \neq \emptyset \\ \text{if and only if } & \forall v \in V.\ x_v \in I_v\end{aligned}$$

Indeed, if $\forall v \in V.\ x_v \in I_v$, then the mapping $f(v) = x_v$ is a solution to instance $\mathcal{I}$. Here again, $f(v) \in \uparrow x_v \cap I_v$ by construction. Moreover, it is an embedding:

- if $v_1 \leq v_2$, then $\uparrow x_{v_2} \subseteq \uparrow x_{v_1}$, *i.e.* $x_{v_1} \leq x_{v_2}$.
- if $v_1 \not\leq v_2$, then $\uparrow x_{v_2} \not\subseteq \uparrow x_{v_1}$, *i.e.* $x_{v_1} \not\leq x_{v_2}$.

Now, to reduce an unresolved instance to resolved instances as in the previous section, we need to handle pairs of elements of V such that $v_1 \not\leq v_2$. This was not the case with the Partial-Embeddability problem since monotone mappings may lose that information. Observe that given a pair $(v_1, v_2) \in V^2$ such that $v_1 \not\leq v_2$, and $U_{v_2} \subseteq U_{v_1}$, if U_{v_2} is a filter (say $U_{v_2} = \uparrow x_2$) then we can update U_{v_1} with $U_{v_1} \cap \complement(\downarrow x_2)$. This operation is sound and complete. Indeed, if the original instance has a solution f, then $f(v_2) \geq x_2$. Therefore, if $f(v_1) \in \downarrow x_2$, then $f(v_1) \leq f(v_2)$, and f would not be a solution. Thus, $f(v_1) \in U_{v_1} \cap \complement(\downarrow x_2)$.

At this point, one can see how this affects the rest of the proof: while in the case of partial-embeddability, we could update upward-closed sets in a certain order on V that was ensuring termination, here there could be a pair $(v_1, v_2) \in V^2$ such that $v_1 \not\leq v_2 \not\leq v_1$ for which the sequence of updates described above never reaches a resolved instance. This suspicion will be confirmed in the next chapter, where the following is proved:

Theorem 10.3.1. *For $(X, \leq) = (A^*, \leq_*)$, where $(A, =)$ is a two-symbol alphabet and $\leq_*$ is the Higman quasi-ordering introduced in Section 6.1: $\Sigma_1(A^*, \leq, \mathtt{A}^*)$ is undecidable.*

This result contrasts with the NP-completeness of $\Sigma_1(A^*, \leq_*)$, proved in [54]. Note that more generally, the decidability of $\Sigma_1(X, \leq)$ reduces to the embeddability problem without constraints, that is given a finite quasi-ordering, does it embed in $(X, \leq)$? This problem in the case of $(\mathbb{N}^k, \leq_\times)$ is called *the dimension problem* [63]. In the case of $(A^*, \leq_*)$ for a finite alphabet A, every instance is positive [54].

Chapter 11

First-Order Logic over the Subword Ordering

11.1 Undecidability of $\Sigma_1(A^*, \leq_*, c_1, \dots)$

This section is dedicated to the proof of undecidability of the existential fragment of $\mathsf{FO}(A^*, \leq_*, \mathtt{A}^*)$ where $A = \{a, b\}$ is a two-symbol alphabet, *i.e.* ordered with equality.

More precisely we prove the following:

Theorem 11.1.1. *For each recursively enumerable set $S \subseteq \mathbb{N}$, there is a Σ_1 formula φ over the structure $\mathsf{FO}(A^*, \leq_*, \mathtt{A}^*)$ with one free variable such that $[\![\varphi]\!] = \{a^k \mid k \in S\}$. In particular, $\Sigma_1(A^*, \leq_*, \mathtt{A}^*)$ is undecidable.*

To prove Theorem 11.1.1, we use the following result on recursively enumerable sets:

Theorem 11.1.2 ([64])**.** *Let $S \subseteq \mathbb{N}$ be a recursively enumerable set. Then there is a finite set of variables $\{x_0, \dots, x_m\}$ and a finite set E of equations, each of the form*

$$x_i = x_j + x_k \qquad\qquad x_i = x_j \cdot x_k \qquad\qquad x_i = 1$$

with $i, j, k \in [0, m]$, such that

$$S = \{y_0 \in \mathbb{N} \mid \exists y_1, \dots, y_m \in \mathbb{N} \colon (y_0, \dots, y_m) \text{ satisfies } E\}.$$

Proof. Of Theorem 11.1.1

The proof consists in building more and more complex predicates that can be expressed in the existential fragment of $\mathsf{FO}(A^*, \leq_*, \mathtt{A}^*)$. The new predicates we express are described in the meta language of mathematics. For each of them, we provide an existential first-order formula in our logic extended with the predicates built so far. Recall that $A = \{a, b\}$. As in the rest of the manuscript, elements in A^* are denoted $\boldsymbol{u}, \boldsymbol{v}, \dots$ in bold font. For variables in the formulas, we use letters x, y, z, u, v, w. When proving the correctness of the formulas we provide, we identify a variable and its valuation, infringing the previous rule.

The following predicates can be expressed in $\Sigma_1(A^*, \leq_*, \mathtt{A}^*)$:

1. Simple languages membership: $x \in \boldsymbol{u}a^*\boldsymbol{v}b^*\boldsymbol{w}$, or simply $x \in \boldsymbol{u}a^*\boldsymbol{v}$.

   In the rest of the proof, we will write $\exists x \in L$ for several languages L of the above form, for some $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in A^*$ and $a, b \in A$. This can indeed be expressed in our logic since we have seen that we can express membership in upward-closed sets and downward-closed sets, and $\boldsymbol{u}a^*\boldsymbol{v}b^*\boldsymbol{w} = \uparrow(\boldsymbol{uvw}) \cap (\downarrow \boldsymbol{u})a^*(\downarrow \boldsymbol{v})b^*(\downarrow \boldsymbol{w})$ (see Section 6.1 for the structure of downward-closed sets of $(A^*, \leq_*)$). Similarly, $\boldsymbol{u}a^*\boldsymbol{v} = \uparrow(\boldsymbol{uv}) \cap (\downarrow \boldsymbol{u})a^*(\downarrow \boldsymbol{v})$.

2. Occurrence comparison (strict): $|u|_a < |v|_a$.
   Recall $|u|_a$ denotes the number of occurrences of a in u.

   $$\exists x \in a^* \colon x \leq_* v \wedge x \not\leq_* u.$$

   The correctness of this expression follows from the observation that for $x \in a^*$, $x \leq_* u$ is equivalent to $|x| \leq |u|_a$.

3. Successor (weak version 1): $\exists n \colon u = a^n \wedge v = a^{n-1}b$.

   $$u \in aaa^* \wedge v \in a^*b \wedge \exists x \in a^*baa \colon |v|_a < |u|_a \wedge v \not\leq_* x \wedge u \leq_* x.$$

   Note that the above formula is actually equivalent to "$\exists n \geq 2 \colon u = a^n \wedge v = a^{n-1}b$", from which it is not difficult to build a formula for the actual predicate we want to express.

   Correctness:
   ($\Rightarrow$) If $u = a^n$ and $v = a^{n-1}b$ for some $n \geq 2$, then the formula is satisfied with $x = a^{n-2}baa$.

   ($\Leftarrow$) Conversely, suppose the formula is satisfied with $u = a^n$, $x = a^\ell baa$ and $v = a^m b$ for some $\ell, m, n \in \mathbb{N}$. Then $|v|_a < |u|_a \wedge v \not\leq_* x \wedge u \leq_* x$ translates as $m < n \wedge \ell < m \wedge n \leq \ell + 2$, *i.e.* $\ell < m < n \leq \ell + 2$ which implies $n = m + 1 = \ell + 2$.

4. Letter occurrence comparison (equality, weak version): $u, v \in A^*b \wedge |u|_a = |v|_a$.

   $$\exists x \in a^* \colon \exists y \in a^*b \colon \left[\exists n \colon x = a^n \wedge y = a^{n-1}b\right]$$
   $$\wedge\, y \leq_* u \;\wedge\; y \leq_* v \;\wedge\; x \not\leq_* u \;\wedge\; x \not\leq_* v.$$

   Correctness:
   ($\Rightarrow$) If $u, v \in A^*b$ with $|u|_a = |v|_a$, then the formula is satisfied with $n = |u|_a + 1$.

   ($\Leftarrow$) Suppose the formula is satisfied. Then $a^{n-1}b \leq_* u$ and $a^n \not\leq_* u$ together imply $|u|_a = n - 1$. Moreover, if u ended in a, then $a^{n-1}b \leq_* u$ would entail $a^n \leq_* u$, which is not the case. Since $|u| \geq 1$, we therefore have $u \in A^*b$. By symmetry, we have $|v|_a = n - 1$ and $v \in A^*b$. Hence, $|u|_a = n - 1 = |v|_a$.

5. Successor (weak version 2): $\exists n \colon u = aaba^nb \wedge v = aba^{n+1}b \wedge w = ba^{n+2}b$.

   $$u \in aaba^*b \wedge v \in aba^*b \wedge w \in ba^*b$$
   $$\wedge\; [u, v, w \in \{a, b\}^*b \wedge |u|_a = |v|_a = |w|_a]\,.$$

6. Successor (weak version 3): $\exists n\colon u = ba^n b \wedge v = ba^{n+1}b$.

$$\exists x, y, z\colon \left[\exists m\colon x = aaba^m b \wedge y = aba^{m+1}b \wedge z = ba^{m+2}b\right] \\ \wedge\, u, v \in ba^*b \;\wedge\; u \le_* y \;\wedge\; u \not\le_* x \;\wedge\; v \le_* z \;\wedge\; v \not\le_* y.$$

   Again, observe that the formula only works for $n \geq 1$.

   Correctness:
   ($\Rightarrow$) If $u = ba^n b$ and $v = ba^{n+1}b$ for some $n \geq 1$, then the formula is satisfied with $m = n - 1$.

   ($\Leftarrow$) Suppose the formula is satisfied for $u = ba^k b$ and $v = ba^\ell b$. Then $u \le_* y \wedge u \not\le_* x$ imply $k = m + 1$; and $v \le_* z \wedge v \not\le_* y$ imply $\ell = m + 2$

7. Successor: $\exists n\colon u = a^n \wedge v = a^{n+1}$.

$$\exists x, y, z\colon \left[\exists m\colon x = ba^m b \wedge y = ba^{m+1}b\right] \\ \wedge \left[\exists k\colon y = ba^k b \wedge z = ba^{k+1}b\right] \\ \wedge\, u, v \in a^* \;\wedge\; u \le_* y \;\wedge\; u \not\le_* x \;\wedge\; v \le_* z \;\wedge\; v \not\le_* y.$$

   Correctness of the above formula (for $n \geq 1$) is similar to the previous one. Note that because of the word y, k has to be equal to $m + 1$.

8. Occurrence comparison (equality): $|u|_a = |v|_a$.

$$\exists x, y\colon \left[\exists n\colon x = a^n \wedge y = a^{n+1}\right] \wedge x \le_* u \wedge y \not\le_* u \wedge x \le_* v \wedge y \not\le_* y$$

   As for the second predicate, correctness relies on the equivalence
   $x \le_* u \Leftrightarrow |x| \leq |u|_a$ for $x \in a^*$.

9. Unary concatenation (weak version): $u \in a^* \wedge v = bu$.

$$u \in a^* \wedge v \in ba^* \wedge |v|_a = |u|_a.$$

   The predicate $u \in a^* \wedge v = ub$ is similarly expressible.

10. Addition: $|w|_a = |u|_a + |v|_a$.

$$\exists x, y \in a^*\colon |x|_a = |u|_a \wedge |y|_a = |v|_a \\ \wedge \exists z \in a^*ba^*\colon\; xb \le_* z \wedge xab \not\le_* z \wedge by \le_* z \wedge bya \not\le_* z \\ \wedge |w|_a = |z|_a$$

   Note that we can define xa and ya thanks to Successor and xb, $(xa)b$, by and $b(ya)$ thanks to Unary concatenation (Items 7 and 9).
   Correctness:
   ($\Rightarrow$) Obvious.

   ($\Leftarrow$) the constraints on the second line enforce that $z = xby$ and hence $|z|_a = |x|_a + |y|_a = |u|_a + |v|_a$.

11. Longest unary suffix: v is the longest suffix of u which is in a^*, which we will later denote by $v = \mathrm{ls}(u, a)$.

$$v \in a^* \wedge \exists x \in b^*a^* \colon \exists y \in b^*a^* \colon$$
$$|x|_b = |y|_b = |u|_b \ \wedge\ |y|_a = |x|_a + 1$$
$$\wedge\ x \leq_* u \ \wedge\ y \not\leq_* u \ \wedge\ |v|_a = |x|_a.$$

Correctness: there are two cases: either $u \in a^*$, or $u = u'ba^n$ for some $u' \in A^*$ and $n \in \mathbb{N}$. The first case is left to the reader.

($\Rightarrow$) If $v = a^n$, then the formula is satisfied with $x = b^{|u'|_b}ba^n$ and $y = xa$.

($\Leftarrow$) Suppose the formula is satisfied, then $v = a^m, x = b^k a^m, y = b^k a^{m+1}$ for $k = |u|_b$ and for some $m \in \mathbb{N}$. Moreover, since $x \leq_* u$, $m \leq n$. And since $y \not\leq_* u$, $m + 1 \not\leq n$, from which we derive $m = n$, and $v = a^n$.

12. Unary concatenation: $v \in a^* \wedge w = uv$.

$$\begin{aligned}
&v \in a^* \wedge \\
\wedge\ &\exists x, y \in a^* : x = \mathrm{ls}(u, a) \ \wedge\ y = \mathrm{ls}(w, a) \\
\wedge\ &|w|_b = |u|_b \ \wedge\ u \leq_* w && (11.1)\\
\wedge\ &|y|_a = |x|_a + |v|_a \ \wedge\ |w|_a = |u|_a + |v|_a && (11.2)
\end{aligned}$$

Correctness:
($\Rightarrow$) If $v \in a^*$ and $w = uv$, then the formula is satisfied with $x = \mathrm{ls}(u, a)$ and $y = \mathrm{ls}(w, a)$.

($\Leftarrow$) Let $v = a^n$, $x = a^p$ and $y = a^q$. The implication being clear if $|u|_b = 0$ or $|w|_b = 0$, we write $u = u'ba^k$ and $w = w'ba^\ell$. The second line implies that $k = p$ and $\ell = q$. Moreover, $y = xv$ (forth line), and thus $w = w'bxv$. Hence, $u \leq_* w$ implies $u \leq_* w'bx$. Thus, w and uv have the same number of occurrences of both a and b, and one is subword of the other: they are therefore equal.

13. Perfect alternation: $u \in (ab)^*$.

$$\exists v \colon v = uab \wedge v = abu$$

Note that the equation $v = uab$ can be obtained using twice Item 12.

Correctness:
($\Rightarrow$) Obvious.

($\Leftarrow$) Assume $abu = uab$. If $u = \epsilon$, then $u \in (ab)^*$. Otherwise, $u = abu'$ for some $u' \in A^*$, and the equation becomes $ababu' = abu'ab$, which is equivalent to $abu' = u'ab$. By induction, we can prove that $u \in (ab)^*$.

14. Occurrence comparison of different letters: $|u|_a = |v|_b$.

$$\exists x \in (ab)^* \colon |u|_a = |x|_a \wedge |v|_b = |x|_b.$$

15. Multiplication: $\exists m, n\colon u = a^n \wedge v = a^m \wedge w = a^{m\cdot n}$.

$$\begin{aligned} & u, v, w \in a^* \\ \wedge\ \exists x\colon [\exists y, z\colon\ y = bu\ \wedge\ z = yx\ \wedge\ z = xy] & \\ \wedge\ |x|_b = |v|_a\ \wedge\ |w|_a = |x|_a. & \end{aligned}$$

Here again $y = bu$, $z = yx$ and $z = xy$ results from several applications of Item 12.

Correctness:
($\Rightarrow$) The formula is satisfied with $x = (bu)^m$.

($\Leftarrow$) The conditions in brackets require $(bu)x = x(bu)$. As in Item 13, a simple induction proves that this implies $x \in (bu)^*$. If $u = a^n$ and $v = a^m$, then the condition $|x|_b = |v|_a$ entails $x = (bu)^m$. Finally, $|w|_a = |x|_a = |u|_a \cdot m = n \cdot m$.

16. Recursively Enumerable sets: $\varphi_S(u) = \exists n \in S : u = a^n$, for any set $S \subseteq \mathbb{N}$ which is recursively enumerable.

We use the fact that every recursively enumerable set of natural numbers is Diophantine. Applying Theorem 11.1.2 to S yields a finite set E of equations over the variables $\{x_0, \ldots, x_m\}$. The formula φ_S is of the form

$$\exists x_1, x_2, \ldots, x_m \in a^*\colon \psi,$$

where ψ is a conjunction of the following formulas:

- for each equation $x_i = 1$, we add $x_i = a$;
- for each equation $x_i = x_j + x_k$, we add a formula expressing $|x_i|_a = |x_j|_a + |x_k|_a$,
- for each equation $x_i = x_j \cdot x_k$, we add a formula expressing $x_i = a^{|x_j| \cdot |x_k|}$.
Then we clearly have $[\![\varphi_S]\!] = \{a^k \mid k \in S\}$. □

11.2 Alternation Bounded Fragments of $\mathsf{FO}(A^*, \leq_*, \ldots)$

In this section, we refine the usual Σ_i fragments ($i \in \mathbb{N}$) with the notion of *letter alternation*. A language $L \subseteq A^*$ over some alphabet $A = \{a_1, \ldots, a_n\}$ is *alternation-bounded* if $L \subseteq (a_1^* \cdots a_n^*)^\ell$ for some $\ell \in \mathbb{N}$. Intuitively, the number of alternations between two distinct letters of a word of L is bounded by $\ell \cdot n$. Equivalently, the number of factors of the form ab for $a, b \in A$, $a \neq b$ is bounded by $\ell \cdot n$.

The motivation behind this notion is the observation that if all variables of a Σ_1 formula are restricted to belong to an alternation-bounded language, then the validity of such formulas reduces to Presburger arithmetic, the first-order theory of natural numbers with addition. Indeed, a word of $(a_1^* \cdots a_n^*)^\ell$ can be encoded using $\ell \cdot n$ integer variables, one for each maximal factor of the form a^*, for $a \in A$. It is thus of interest to investigate the effect of letter alternation on decidability: while Σ_i bounds the quantifier alternation of a formula, the fragment $\Sigma_{i,j}$ also bounds the number of variable whose (letter) alternation is not bounded.

Before defining the $\Sigma_{i,j}$ fragments, we need to formalise the possibility to restrict quantified variables to alternation-bounded languages. In this section, we will therefore use first-order logic formulas over the following extended syntax:

$$\begin{aligned}\varphi ::= t \leq t \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \exists x.\, \varphi \mid \forall x.\, \varphi \\ \mid \exists x \in (a_1^* \cdots a_n^*)^\ell.\, \varphi \mid \forall x \in (a_1^* \cdots a_n^*)^\ell.\, \varphi\end{aligned}$$

Note that the language $(a_1^* \cdots a_n^*)^\ell$ is downward-closed, therefore this extended logic is not more expressive than the one we used until now.

Definition 11.2.1. *Given a closed first-order formula φ and a variable $x \in \mathrm{Var}$, we say that x is* alternation-bounded *in φ if all quantifications over x occurring in φ are guarded by a language $(a_1^* \cdots a_n^*)^\ell$ for some ℓ, that is are of the form $Qx \in (a_1^* \cdots a_n^*)^\ell.\, \psi$ for $Q \in \{\forall, \exists\}$. Otherwise, x is said to be* alternation-unbounded*.*

The fragment $\Sigma_{i,j}$ consists of all Σ_i closed formulas with less than j alternation-unbounded variables.

In the next subsection, we formalise the intuition given above, proving that $\Sigma_{1,0}$ is decidable. We then prove the decidability of $\Sigma_{1,1}$ in Section 11.2.2, by reduction to $\Sigma_{1,0}$.

11.2.1 Decidability of $\Sigma_{1,0}$

Theorem 11.2.2. *The $\Sigma_{1,0}$ fragment is decidable.*

We reduce $\Sigma_{1,0}$ to existential Presburger arithmetic, the first-order theory of natural numbers with ordering and addition (but no multiplication). This logic has been proved decidable by Mojzesz Presburger in his Master thesis in 1929.

Let φ be a closed formula in $\Sigma_{1,0}$, and ℓ be the greatest alternation-bound appearing in φ. A word in $\boldsymbol{u} \in (a_1^* \cdots a_n^*)^\ell$ can be uniquely written $\boldsymbol{u} = \prod_{j=1}^{\ell} a_1^{x_1^j} a_2^{x_2^j} \cdots a_n^{x_n^j}$. Therefore, a variable x in φ will be encoded by $\ell \cdot n$ Presburger variables $(x_i^j)_{i\in[n],j\in[\ell]}$ following this decomposition. For variables with alternation bound $k < \ell$, it suffices to add the constraints $x_i^j = 0$ for $i \in [n]$ and $k < j \leq \ell$. In the other direction, given $\boldsymbol{x} = \langle x_1^1, \ldots, x_n^\ell \rangle \in \mathbb{N}^{\ell \cdot n}$, we denote by $w_{\boldsymbol{x}}$ the associated word $w_{\boldsymbol{x}} = \prod_{j=1}^{\ell} a_1^{x_1^j} \cdots a_n^{x_n^j}$.

Since $\leq_*$ is the only predicate in our logic, it suffices to build Presburger formulas for $x \leq_* y$ and $x \not\leq_* y$ with the above encoding of variables (notice that we can also use this encoding for constants). The rest of the reduction is straightforward: we replace every occurrence of $x \leq_* y$ or $x \not\leq_* y$ in φ by the adequate Presburger formula to obtain a Presburger formula which is equivalent to φ.

Proposition 11.2.3. *There are existential Presburger formulas $\psi_{\leq_*}$ and $\psi_{\not\leq_*}$ such that:*

$$\begin{aligned}\psi_{\leq_*}(x_1^1, \ldots, x_n^\ell, y_1^1, \ldots, y_n^\ell) &\iff w_{\boldsymbol{x}} \leq_* w_{\boldsymbol{y}}, \\ \psi_{\not\leq_*}(x_1^1, \ldots, x_n^\ell, y_1^1, \ldots, y_n^\ell) &\iff w_{\boldsymbol{x}} \not\leq_* w_{\boldsymbol{y}}.\end{aligned}$$

Proof. Let $I = [n] \times [\ell]$ and order the pairs $(i,j) \in I$ lexicographically: $(i',j') \preceq (i,j)$ if $j' < j$ or $j = j'$ and $i' < i$. This captures the order of the $a_i^{x_i^j}$ factors in $w_{\boldsymbol{x}}$.

We first define formulas τ and η that use extra free variables $t_{i,j,k}$'s and $e_{i,j,k}$'s for $(i,j,k) \in [n] \times [\ell] \times [\ell] \stackrel{\text{def}}{=} J$:

$$\tau \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_{(i,j,k)\in J} t_{i,j,k} \quad = \begin{cases} 0 & \text{if } e_{i',j',k'} > 0 \text{ for some } (i',j') \preceq (i,j) \text{ and } k' > k \\ y_i^k - \sum_{j'=1}^{j-1} e_{i,j',k} & \text{otherwise} \end{cases}$$

$$\eta \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_{(i,j,k)\in J} e_{i,j,k} \quad = \min\left(t_{i,j,k}\ ,\ x_i^j - \sum_{r=1}^{k-1} e_{i,j,r}\right)$$

These expressions define the leftmost embedding of $w_{\boldsymbol{x}}$ into $w_{\boldsymbol{y}}$: the variable $t_{i,j,k}$ describes how many letters from $a_i^{y_i^k}$ are available for embedding the $a_i^{x_i^j}$ factor of $w_{\boldsymbol{x}}$. The variable $e_{i,j,k}$ counts how many of these available letters are actually used for the $a_i^{x_i^j}$ factor in the left-most embedding of w_x into w_y.

We are now ready to define $\psi_{\leq_*}$ and $\psi_{\not\leq_*}$:

$$\psi_{\leq_*} \stackrel{\text{def}}{\Leftrightarrow} \exists (t_{i,j,k})_J \exists (e_{i,j,k})_J : \tau \wedge \eta \wedge \bigwedge_{(i,j)\in I} \left(x_i^j \leq \sum_{k=1}^{\ell} e_{i,j,k} \right)$$

$$\psi_{\not\leq_*} \stackrel{\text{def}}{\Leftrightarrow} \exists (t_{i,j,k})_J \exists (e_{i,j,k})_J : \tau \wedge \eta \wedge \bigvee_{(i,j)\in I} \left(x_i^j > \sum_{k=1}^{\ell} e_{i,j,k} \right)$$

Since formulas τ and η are inductive equations that uniquely define the values of $t_{i,j,k}$ and $e_{i,j,k}$ as functions of the x and y vectors, ψ is equivalent to the negation of φ (quantifying universally or existentially on the $t_{i,j,k}$ and $e_{i,j,k}$ yields equivalent formulas). Moreover, φ expresses that there is enough room to embed each factor $a_i^{x_i^j}$ in $w_{\boldsymbol{y}}$, i.e., that $w_{\boldsymbol{x}} \leq_* w_{\boldsymbol{y}}$ as claimed. □

11.2.2 Decidability of $\Sigma_{1,1}$

Theorem 11.2.4. *The $\Sigma_{1,1}$ fragment is decidable.*

Decidability is obtained by reduction to $\Sigma_{1,0}$. In the fashion of a quantifier elimination procedure, we show that formulas of the form $\exists t \colon \varphi$ where $\varphi \in \Sigma_{1,0}$ and t is the only alternation-unbounded variable, are equivalent to some computable $\Sigma_{1,0}$ formula (next proposition). To compute an equivalent $\Sigma_{1,0}$ formula to any $\Sigma_{1,1}$ closed formula ψ, it then suffices to proceed by induction.

Proposition 11.2.5. *Let φ be a $\Sigma_{1,0}$ formula with a single free variable t. Then $\exists t \colon \varphi$ is equivalent to $\exists t \in (a_1^* \cdots a_n^*)^p$ for some computable $p \in \mathbb{N}$.*

Proof. To prove this proposition, we exhibit a natural number p, computable from φ, such that for any word $\boldsymbol{u} \in A^*$, if $\boldsymbol{u} \notin (a_1^* \cdots a_n^*)^p$ and $\boldsymbol{u} \in [\![\varphi]\!]$, then there exists $\boldsymbol{v} \in [\![\varphi]\!]$ whose alternation is strictly smaller than $\boldsymbol{u}$'s (but not necessarily smaller than p). Thus, starting from any $\boldsymbol{u} \in [\![\varphi]\!]$, and by iterating the aforementioned property, we eventually get a solution to φ which is of alternation at most p.

However, the alternation decreases in each step are so small that we need a finer notion of alternation to prove that the alternation strictly decrease at each step. Any word $\boldsymbol{u} \in A^*$ can be factored into *blocks* of repeating letters, *i.e.* $\boldsymbol{u} = \prod_{i=1}^{k} {a_i}^{\ell_i}$ with $\ell_i > 0$ and $a_i \neq a_{i+1}$ for all i. By an *a-block of $\boldsymbol{u}$*, we mean an occurrence of a factor $a_i^{\ell_i}$ with $a_i = a$. Subsequently, we use the number of blocks of a word as a measure of alternation of the word. Note that if a word has k blocks, then it belongs to $(a_1^* \cdots a_n^*)^k$. Conversely, a word in $(a_1^* \cdots a_n^*)^\ell$ has at most $\ell \cdot n$ blocks. Therefore, alternation-bounded languages are the same for the two notions.

Without loss of generality, we assume φ to be in prenex form: $\varphi = \exists z_1 \ldots \exists z_k \colon \psi$. Further assume that ψ has no sub-formula of the form $t \bowtie \mathtt{u}$ for any constant $\mathtt{u}$. This can be assumed since we can always introduce a new alternation-bounded variable which is equal to $\mathtt{u}$. Let $(\boldsymbol{t}, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \in [\![\psi]\!]$ be a solution to ψ. Let ℓ be the maximal number of blocks of the words $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_k$. We now show that if $\boldsymbol{t}$ has more than $p \stackrel{\text{def}}{=} k \cdot \ell + n$ blocks, then there exists $\boldsymbol{t}'$ such that $(\boldsymbol{t}', \boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \in [\![\psi]\!]$ and $\boldsymbol{t}'$ has strictly fewer blocks than $\boldsymbol{t}$.

Given $\boldsymbol{u} \in A^*$, we write $\operatorname{Im} \boldsymbol{u}$ for the image of the left-most embedding of $\boldsymbol{u}$ into $\boldsymbol{t}$. This is a set of positions in $\boldsymbol{t}$ and, in case $\boldsymbol{u} \not\leq_* \boldsymbol{t}$, these positions only account for the longest prefix of $\boldsymbol{u}$ that can be embedded in $\boldsymbol{t}$. In particular, $|\operatorname{Im} \boldsymbol{u}| = |\boldsymbol{u}|$ if and only if $\boldsymbol{u} \leq_* \boldsymbol{t}$ (and $|\operatorname{Im} \boldsymbol{u}| < |\boldsymbol{u}|$ otherwise).

Formally, if $\boldsymbol{u} = u_1 \cdots u_n$ and $\boldsymbol{t} = t_1 \cdots t_m$, define the left-most embedding f as follows: $f(1)$ is the smallest natural number k in $[1, m]$ such that $u_1 = t_k$, if it exists, $f(1)$ is left undefined otherwise. For $i \in [2, n]$, if $f(j)$ as been defined up to $i - 1$, let $f(i)$ be the smallest k in $[f(i-1), m]$ such that $u_i = t_k$, if it exists. In any other case, $f(i)$ is left undefined. In the end, $\operatorname{Im} \boldsymbol{u}$ is the set of all indexes $f(1), \ldots, f(n)$ that have been defined.

Let $\boldsymbol{b}_0$ be an a-block of $\boldsymbol{t}$. This block is said to be *irreducible* if and only if:

1. either it is the last, i.e. right-most, a-block of $\boldsymbol{t}$,
2. or writing $\boldsymbol{t}$ under the form $\boldsymbol{t} = \boldsymbol{t}_0 \boldsymbol{b}_0 \boldsymbol{t}_1 \boldsymbol{b}_1 \boldsymbol{t}_2$ where $\boldsymbol{b}_1$ is the next a-block, i.e. $a \notin \boldsymbol{t}_1$, one of the following holds:
   - there is some $i \in [k]$ such that:
     $$\boldsymbol{z}_i \leq_* \boldsymbol{t} \quad \text{and} \quad \boldsymbol{b}_0 \cap \operatorname{Im} \boldsymbol{z}_i \neq \emptyset \quad \text{and} \quad \boldsymbol{t}_1 \cap \operatorname{Im} \boldsymbol{z}_i \neq \emptyset.$$
   - there is $i \in [k]$ such that:
     $$\boldsymbol{z}_i \not\leq_* \boldsymbol{t} \quad \text{and} \quad \boldsymbol{b}_0 \cap \operatorname{Im} \boldsymbol{z}_i = \emptyset \quad \text{and} \quad \boldsymbol{t}_1 \cap \operatorname{Im} \boldsymbol{z}_i \neq \emptyset \quad \text{and} \quad \boldsymbol{b}_1 \cap \operatorname{Im} \boldsymbol{z}_i \neq \emptyset.$$

Otherwise $\boldsymbol{b}_0$ is said to be *reducible*.

The whole point of reducible blocks is that they can be swapped to the right: if $\boldsymbol{b}_0$ is a reducible a-block of $\boldsymbol{t}$, then $\boldsymbol{t}$ can be decomposed $\boldsymbol{t} = \boldsymbol{t}_0 \boldsymbol{b}_0 \boldsymbol{t}_1 \boldsymbol{b}_1 \boldsymbol{t}_2$ as above, and the word $\boldsymbol{t}' = \boldsymbol{t}_0 \boldsymbol{t}_1 \boldsymbol{b}_0 \boldsymbol{b}_1 \boldsymbol{t}_2$ satisfies the conditions we want. Indeed, since $\boldsymbol{b}_0$ and $\boldsymbol{b}_1$ are both blocks of the same letter, $\boldsymbol{b}_0 \boldsymbol{b}_1$ is now a single block of $\boldsymbol{t}'$, which means that $\boldsymbol{t}'$ has at least one less block than $\boldsymbol{t}$. Moreover, $(\boldsymbol{t}', \boldsymbol{z}_1, \ldots, \boldsymbol{z}_k) \in [\![\psi]\!]$: we show that for any $i \in [k]$, $\boldsymbol{z}_i \leq_* \boldsymbol{t}$ if and only if $\boldsymbol{z}_i \leq_* \boldsymbol{t}'$.

- Let $i \in [k]$ such that $\boldsymbol{z}_i \leq_* \boldsymbol{t}$, there is a unique decomposition $\boldsymbol{z}_i = \boldsymbol{u}_0\boldsymbol{u}_1\boldsymbol{u}_2\boldsymbol{u}_3\boldsymbol{u}_4$ of $\boldsymbol{z}_i$ such that $\operatorname{Im} u_0 \subseteq \boldsymbol{t}_0$, $\operatorname{Im} u_1 \subseteq \boldsymbol{b}_0$, $\operatorname{Im} u_2 \subseteq \boldsymbol{t}_1$, $\operatorname{Im} u_3 \subseteq \boldsymbol{b}_1$ and $\operatorname{Im} u_4 \subseteq \boldsymbol{t}_2$. Since $\boldsymbol{b}_0$ is reducible, one of $\operatorname{Im} \boldsymbol{u}_1$ or $\operatorname{Im} \boldsymbol{u}_2$ is empty. Thus one of $\boldsymbol{u}_1$ or $\boldsymbol{u}_2$ is the empty word, entailing $\boldsymbol{z}_i \leq_* \boldsymbol{t}'$.
- Let $i \in [k]$ such that $\boldsymbol{z}_i \not\leq_* \boldsymbol{t}$. Assume, by way of contradiction, that $\boldsymbol{z}_i \leq_* \boldsymbol{t}'$. Let $\boldsymbol{u}_1$ be the maximal prefix of $\boldsymbol{z}_i$ that embeds into $\boldsymbol{t}$. We proceed to show that $\boldsymbol{b}_0$ is irreducible (a contradiction).
  - Firstly, $\boldsymbol{b}_0 \cap \operatorname{Im} \boldsymbol{u}_1 = \emptyset$. Otherwise, since $a \notin \boldsymbol{t}_1$, the left-most embedding of $\boldsymbol{u}_1$ into $\boldsymbol{t}' = \boldsymbol{t}_0\boldsymbol{t}_1\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{t}_2$ does not use t_1 at all and we would have $\boldsymbol{z}_i \leq_* \boldsymbol{t}_0\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{t}_2 \leq_* \boldsymbol{t}$.
  - Secondly, $\boldsymbol{t}_1 \cap \operatorname{Im} \boldsymbol{u}_1$ is not empty. If it were, since $a \notin t_1$, the left-most embedding of $\boldsymbol{u}_1$ into $\boldsymbol{t}_0\boldsymbol{t}_1\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{t}_2$ would not use $\boldsymbol{t}_1$ and again we would have $\boldsymbol{z}_i \leq_* \boldsymbol{t}_0\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{t}_2 \leq_* \boldsymbol{t}$.
  - Lastly, $\boldsymbol{b}_1 \cap \operatorname{Im} \boldsymbol{u}_1 \neq \emptyset$. Otherwise, the already established condition $\boldsymbol{b}_0 \cap \operatorname{Im} \boldsymbol{u}_1 = \emptyset$ implies that $\boldsymbol{z}_i$ embeds not only in $\boldsymbol{t}'$ but in $\boldsymbol{t}_0\boldsymbol{t}_1\boldsymbol{t}_2$, which is a subword of $\boldsymbol{t}$.

Now to conclude the proof, it remains to show that $\boldsymbol{t}$ indeed has a reducible block. This results from our choice of p: every irreducible block is either a right-most a-block for some a (n possible blocks), or can be associated with a block alternation in some $\boldsymbol{z}_i$ ($\ell \cdot k$ possible blocks). Thus there are at most $\ell \cdot k + n = p$ irreducible blocks in $\boldsymbol{t}$. Since we assumed that $\boldsymbol{t}$ has more than p blocks, $\boldsymbol{t}$ must have some reducible blocks.

□

11.3 Concluding Remarks

Other Results from [62]. In the previous Section, we have introduced new fragments $\Sigma_{i,j}$ of the logic $\mathsf{FO}(A^*, \leq_*, \mathtt{A}^*)$ that refine the usual fragments Σ_i. We proved the decidability of $\Sigma_{1,0}$ by polynomial-time reduction to the existential fragment of Presburger, hence proving a NP upper bound for this fragment. The two problems are actually inter-reducible, settling the NP-completeness of $\Sigma_{1,0}$. The inter-reducibility further carries over any quantifier rank, and $\Sigma_{i,0}$ is inter-reducible with the Σ_i fragment of Presburger Arithmetic (see [62] for details). Recent results of Haase [65] on these fragments then settles the complexity of $\Sigma_{i,0}$ presented in Table 11.1. The notation Σ_n^{EXP} used in this table denotes the n-th level of the weak EXP hierarchy, which lies between NEXP and $\mathsf{EXPSPACE}$ [66, 67].

We then provided a polynomial-time reduction from $\Sigma_{1,1}$ to $\Sigma_{1,0}$ from which we derive that $\Sigma_{1,1}$ is NP-complete as well. A careful analysis of the proof of undecidability of Σ_1 in Section 11.1 actually reveals that already $\Sigma_{1,3}$ is undecidable. The gap is closed in [62]: $\Sigma_{1,2}$ is also decidable. However, only a NEXP upper bound is known, for the same NP lower bound. The exact complexity of this fragment remains an open problem.

$\Sigma_{i,j}$	0	1	2	3
1	NP	NP	in NEXP	U
$i \geq 2$	$\Sigma^{\mathsf{EXP}}_{i-1}$	U	U	U

Table 11.1: The cell in row i and column j shows the decidability/complexity of the fragment $\Sigma_{i,j}$, where U denotes undecidability.

All other fragments are undecidable, which follows either from the aforementioned undecidability of $\Sigma_{1,3}$ or from the undecidability of $\Sigma_{2,1}$, which is proved in [62]. These results are compiled in Table 11.1.

Perspectives. In Section 11.1, we proved that $\Sigma_1(A^*, \leq_*, \mathtt{A}^*)$ was already undecidable for A a two-symbol alphabet. This implies the undecidability for any alphabet of two or more symbols. We actually proved that it also implies the undecidability of $\Sigma_1(X^*, \leq_*, \mathtt{X}^*)$ for any finite ordering $(X, \leq)$ in which there exists an incomparable pair of elements $a \perp b$.

Therefore, the only finite orderings for which the decidability status of $\Sigma_1(X^*, \leq_*, \mathtt{X}^*)$ is not settled are the linear orderings. Observe that if X is a singleton, $\mathsf{FO}(X^*, \leq_*, \mathtt{X}^*)$ is the theory of integers with ordering, which is PSPACE-complete [68, 69]. Otherwise, X embeds the linearly-ordered set $(\{0,1\}, 0 \leq 1)$. We conjecture that $\Sigma_1(\{0,1\}^*, \leq_*, \{\mathtt{0},\mathtt{1}\}^*)$ is undecidable, which would imply that $\Sigma_1(X^*, \leq_*, \mathtt{X}^*)$ is undecidable for any order $(X, \leq)$ with $|X| \geq 2$. However, we could only prove that $\Sigma_2(\{0,1\}^*, \leq_*, \{\mathtt{0},\mathtt{1}\}^*)$ is undecidable, which already means that $\Sigma_2(X^*, \leq_*, \mathtt{X}^*)$ is undecidable for any partial order $(X, \leq)$ with $|X| \geq 2$.

From the perspective of Chapter 10, Theorem 11.1.1 shows that $\Sigma_1(X, \leq, \mathtt{X})$ cannot be decided in general for an arbitrary ideally effective WQO $(X, \leq)$. In order to find an extra sufficient criterion for an ideally effective WQO to have a decidable existential fragment, a promising angle would be to study other order constructions that preserve ideal effectiveness. This would once again result in an algebra of WQOs whose existential fragment is decidable. For instance, the Cartesian product preserves Σ_1 decidability. Another construction that would in addition be interesting in practice would be finite multisets. Since $(A^{\circledast}, \leq_{\mathrm{emb}})$ when $(A, =)$ is a finite alphabet is isomorphic to $(\mathbb{N}^k, \leq_\times)$, its first-order theory is decidable (using Presburger arithmetic). What about $\Sigma_1(\mathbb{N}^{\circledast}, \leq_{\mathrm{emb}})$ for instance ?

In the case some constructions turn out to preserve Σ_1 decidability, one can wonder about higher quantification alternation. For instance, is $\Sigma_2(X \times Y, \leq_\times, \mathtt{X}\times\mathtt{Y})$ decidable whenever $\Sigma_2(X, \leq_X, \mathtt{X})$ and $\Sigma_2(Y, \leq_Y, \mathtt{Y})$ are ?

Finally, what about the converse implication ? The natural way to represent downward-closed sets in logic is with "excluded minors", and as shown in Section 8.3, this representation may fail to distinguish ideals. However, directedness is a Π_2 formula, and therefore if $\Pi_2(X, \leq, \mathtt{X})$ is decidable, then it is decidable whether $[\![\varphi]\!]$ is an ideal. But otherwise, this is unlikely that decidability of the existential fragment, or positive existential fragment, imply ideal effectiveness in a general way. In particular because our definition requires downward-closed sets to be represented as finite union of ideals.

This raises the question: can we find a non effective WQO whose first-order theory is decidable ? Or simply its existential fragment, or even positive existential fragment ?

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Résumé

Avec des motivations venant du domaine de la Vérification, nous définissons une notion de WQO effectifs pour lesquels il est possible de représenter les ensembles clos et de calculer les principales opérations ensemblistes sur ces représentations. Dans une première partie, nous montrons que de nombreuses constructions naturelles sur les WQO préservent notre notion d'effectivité, prouvant ainsi que la plupart des WQOs utilisés en pratique sont effectifs. Cette partie est basée sur un article non publié dont Jean Goubault-Larrecq, Narayan Kumar, Prateek Karandikar et Philippe Schnoebelen sont co-auteurs.

Dans une seconde partie, nous étudions les conséquences qu'a notre notion sur la logique du première ordre interprété sur un WQO. Bien que le fragment existentiel positif soit décidable pour tous les WQOs effectif, les perspectives de généralisation sont limitées par le résultat suivant: le fragment existentiel de la logique du première ordre sur les mots finis, ordonnés par plongement, est déjà indécidable. Ce résultat a été publié à LICS 2017 avec Philippe Schnoebelen et Georg Zetzsche.

Abstract

With motivations coming from Verification, we define a notion of effective WQO for which it is possible to represent closed subsets and to compute basic set-operations on these representations. In a first part, we show that many of the natural constructions that preserve WQOs also preserve our notion of effectiveness, proving that a large class of commonly used WQOs are effective. This part is based on an unpublished article with Jean Goubault-Larrecq, Narayan Kumar, Prateek Karandikar and Philippe Schnoebelen.

In a second part, we investigate the consequences of our notion on first-order logics over WQOs. Although the positive existential fragment is decidable for any effective WQO, the perspective of extension to larger fragments is hopeless since the existential fragment is already undecidable for the first-order logic over words with the subword ordering. This last result has been published in LICS 2017 with Philippe Schnoebelen and Georg Zetzsche.